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TWISTED HEISENBERG-VIRASORO VERTEX OPERATOR ALGEBRA

Hongyan Guo and Qing Wang
Central China Normal University and Xiamen University, China

Abstract. In this paper, we study a new kind of vertex operator algebras related to twisted Heisenberg-Virasoro algebra. We showed that there exist one-to-one correspondences between the restricted module categories of twisted Heisenberg-Virasoro algebras of rank one and rank two and several different kinds of module categories of their corresponding vertex algebras. We also study in detail the structures of the twisted Heisenberg-Virasoro vertex operator algebra and give a characterization of it as a tensor product of two well-known vertex operator algebras.

1. Introduction

This paper mainly consists of two parts. One is the relationship between (ϕ-coordinated) modules of certain vertex algebras and restricted modules of two Lie algebras, the results are good in that we get equivalence of categories of modules, which may provide new ways of looking at the representation theory of the two Lie algebras as well as the representation theory of the obtained vertex algebras. The other is a fulfilled study of vertex operator algebras we obtained. This kind of vertex operator algebra looks like the form of combining Heisenberg and Virasoro vertex operator algebras. The structure theory and representation theory of vertex operator algebras coming from Heisenberg algebras and Virasoro algebras are well-known and beautiful, so it is inevitable to consider the corresponding theory of our obtained vertex operator algebras.

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The rank one twisted Heisenberg-Virasoro algebra $\mathcal{L}$ was first studied in the paper [2], it is spanned by the elements $L_n, I_n, c_1, c_2, c_3, n \in \mathbb{Z}$, and the Lie bracket is given by (cf. [5])

$$\begin{aligned}[L_m, L_n] &= (m-n)L_{m+n} + \delta_{m+n,0}\frac{m^3-m}{12}c_1, \\ [L_m, I_n] &= -nI_{m+n} - \delta_{m+n,0}(m^2+m)c_2, \\ [I_m, I_n] &= m\delta_{m+n,0}c_3, \quad [\mathcal{L}, c_i] = 0,\ i = 1,2,3.\end{aligned}$$

With $L_n \mapsto -t^{n+1}\frac{d}{dt}$ and $I_n \mapsto t^n$, it is the universal central extension of the Lie algebra of differential operators on a circle of order at most one:

$$\{f(t)\frac{d}{dt} + g(t) \mid f(t), g(t) \in \mathbb{C}[t, t^{-1}]\}.$$

The highest weight modules of $\mathcal{L}$ when c_3 acts in a nonzero way have been studied in the paper [2]. The study for c_3 acts as zero is given in [5]. $\mathcal{L}$ also has its role in the representation theory of toroidal Lie algebra ([6]). Recently, the authors in [3] gave a free field realization of the twisted Heisenberg-Virasoro algebra and study the representation theory of it when c_3 acts as zero using vertex-algebraic methods and screening operators. The representation theory of Heisenberg-Virasoro vertex operator algebras is also related to logarithmic conformal field theory ([4]). In our paper, we study the restricted modules of $\mathcal{L}$ using vertex algebra methods and formal variables, we give a characterization of this type of modules via vertex algebras and corresponding (ϕ-coordinated) modules, where our c_1, c_2, c_3 can act as any complex numbers. And the results are also used to study the irreducible modules of the obtained vertex operator algebras.

In [33], the authors generalized the rank one to rank two case, and call the Lie algebra arising from 2-dimensional torus, which here we denote it by $\mathcal{L}^*$. More precisely, let $A = \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}]$ be the ring of Laurent polynomials in two variables and B be the set of skew derivations of A spanned by the elements of the form

$$E_{m,n} = t_1^m t_2^n (nd_1 - md_2),$$

where $(m,n) \in \mathbb{Z}^2$, and d_1, d_2 are degree derivations of A. Set $L = A \oplus B$. Then L becomes a Lie algebra under the Lie bracket relations

$$\begin{aligned}&[t_1^m t_2^n, t_1^r t_2^s] = 0; \\ &[t_1^m t_2^n, E_{r,s}] = (nr-ms)t_1^{m+r}t_2^{n+s}; \\ &[E_{m,n}, E_{r,s}] = (nr-ms)E_{m+r,n+s}.\end{aligned}$$

Let L' be the derived Lie subalgebra of L. Then L' is perfect and has a universal central extension $\mathcal{L}^*$ with the following Lie bracket relations ([33]):

$$[t_1^m t_2^n, t_1^r t_2^s] = 0; \quad [K_i, \mathcal{L}^*] = 0,\ i = 1, 2, 3, 4;$$
$$[t_1^m t_2^n, E_{r,s}] = (nr - ms)t_1^{m+r} t_2^{n+s} + \delta_{m+r,0}\delta_{n+s,0}(mK_1 + nK_2);$$
$$[E_{m,n}, E_{r,s}] = (nr - ms)E_{m+r,n+s} + \delta_{m+r,0}\delta_{n+s,0}(mK_3 + nK_4).$$

where $(m, n), (r, s) \in \mathbb{Z}^2 \backslash \{(0, 0)\}$, K_1, K_2, K_3, K_4 are central elements.

In this paper, we give an association of the restricted modules of $\mathcal{L}^*$ with ϕ-coordinated modules of corresponding vertex algebra, where again our K_1, K_2, K_3, K_4 can act as arbitrary complex numbers.

In a series of papers, the authors use Lie algebras to construct and study vertex algebras, and they also give the connections between the modules of the Lie algebras and the modules of the corresponding vertex algebras or their likes. For example, the very beginning study of the association of affine and Virasoro algebras with vertex (operator) algebras (cf. [15]), and further studies like [24, 26, 27, 8], etc. Later on, many other Lie algebras, like toroidal Lie algebras, quantum torus Lie algebras, deformed Heisenberg Lie algebras, Lie algebra $\mathfrak{gl}_\infty$, elliptic affine Lie algebra, q-Virasoro algebra and unitary Lie algebra have also been related to vertex algebras or their likes (see [15, 7, 32, 25, 27, 20, 18, 19]).

As for the rank one twisted Heisenberg-Virasoro algebra $\mathcal{L}$, it contains both a Heisenberg subalgebra and a Virasoro subalgebra. In the theory of vertex algebras, we usually write the generating functions of Virasoro algebra and Heisenberg algebra as

$$L(x) = \sum_{n\in\mathbb{Z}} L_n x^{-n-2}, \quad I(x) = \sum_{n\in\mathbb{Z}} I_n x^{-n-1},$$

so we first consider these types of generating functions. After writing the bracket relations in terms of generating functions, we see that the subset which consists of $L(x)$ and $I(x)$, when acting on a restricted module W of $\mathcal{L}$, form a local subset (see [27], cf. [24]). So conceptually, it generates a vertex algebra with W a module under a certain vertex operator operation. And in this case, the explicit vertex algebra we needed is actually an induced module constructed from $\mathcal{L}$. In the process, we observe that when writing the generating functions as the form

$$\widetilde{L}(x) = \sum_{n\in\mathbb{Z}} L_n x^{-n}, \quad \widetilde{I}(x) = \sum_{n\in\mathbb{Z}} I_n x^{-n},$$

the subset that consists of $\widetilde{L}(x)$ and $\widetilde{I}(x)$, when acting on a restricted module W of $\mathcal{L}$, also forms a local subset, but under the vertex operator operation which was introduced through the study of quantum vertex algebras and their corresponding modules ([28, 29]), it generates conceptually a vertex algebra with W being its ϕ-coordinated module. And in this case, we use another Lie algebra which is isomorphic to $\mathcal{L}$ to construct the vertex algebra we needed.

For the rank two twisted Heisenberg-Virasoro algebra $\mathcal{L}^*$, we write the generating functions as

$$T_m(x) = \sum_{n\in\mathbb{Z}} t_1^m t_2^n x^{-n}, \quad E_m(x) = \sum_{n\in\mathbb{Z}} E_{m,n} x^{-n}.$$

Form the subset $U_W = \{\mathbf{1}_W\} \cup \{T_m(x), E_m(x) \mid m \in \mathbb{Z}\}$ with W being a restricted module of $\mathcal{L}^*$, this subset is also a local subset. And under the context of [28] or [29], it generates a vertex algebra with W being its ϕ-coordinated modules. To associate a vertex algebra to $\mathcal{L}^*$ explicitly, we construct a new affine type Lie algebra $\widehat{\mathfrak{L}^*}$ and showed that its induced module $V_{\widehat{\mathfrak{L}^*}}(\ell_{1234}, 0)$ is a vertex algebra, where $\ell_{1234} \in \mathbb{C}$ (see section 4). Furthermore, we prove the correspondence between the restricted modules of $\mathcal{L}^*$ and ϕ-coordinated modules of the vertex algebra $V_{\widehat{\mathfrak{L}^*}}(\ell_{1234}, 0)$.

Beside the equivalence between the categories of these two Lie algebra modules and corresponding vertex algebra modules, we study in detail the new vertex operator algebra $V_{\mathcal{L}}(\ell_{123}, 0)$ obtained in section 2, give the characterization of all the irreducible vertex operator algebra $V_{\mathcal{L}}(\ell_{123}, 0)$-modules, and then we consider the Zhu's algebra, C_2-cofiniteness, rationality, regularity, unitary property of it and its simple descendant, we also consider the commutant of Heisenberg vertex operator algebra in it, in the process, we give a characterization of it as a tensor product of two other vertex operator algebras which are equipped with non standard conformal vectors (See section 3 for detail).

Y. Zhu in [36] constructed an associative algebra $A(V)$ (nowadays it is called Zhu's algebra of V) for a general vertex operator algebra V and established a 1-1 correspondence between irreducible representations of V and irreducible representations of $A(V)$. C_2-cofiniteness, rationality and regularity are three different but closely related notions, and Zhu's algebra plays an important role in studying them, since for V being C_2-cofinite, rational and regular, $A(V)$ must be a finite dimensional algebra (c.f. [36, 12]). The notion of regularity (which deals with the complete reducibility of weak modules) was first introduced in the paper [11], it is a generalization of rationality (which was first introduced in the paper [36] and deals with the complete reducibility of admissible modules). Regularity implies rationality by definition, and it was showed in [31] that any regular vertex operator algebra is C_2-cofinite, C_2-cofiniteness and rationality is equivalent to regularity for CFT type vertex operator algebras has been proved in [1]. For our vertex operator algebra $V_{\mathcal{L}}(\ell_{123}, 0)$, it is closely related to two kinds of vertex operator algebras, as one may expected, Virasoro and Heisenberg vertex operator algebras, their C_2-cofiniteness, rationality and regularity are well-known (c.f. [11, 15, 35]). In section 3, we show in two different ways that the Zhu's algebra of $V_{\mathcal{L}}(\ell_{123}, 0)$ is infinite-dimensional, and its simple descendant also turns out to be infinite-dimensional, which immediately give that our $V_{\mathcal{L}}(\ell_{123}, 0)$ and its simple descendant are not C_2-cofinite, not rational and not regular.

The unitary property of vertex operator algebras has been studied in [9], and most well-known vertex operator algebras turn out to be unitary, our $V_{\mathcal{L}}(\ell_{123}, 0)$ is also proved to be unitary under certain conditions. Unitarity of a vertex operator algebra is important in that it is the first step that one may want to construct conformal nets from vertex operator algebras, where the construction of conformal nets and the construction of vertex operator algebras are expected to be equivalent in the sense that you may get one from the other. The commutant of a vertex subalgebra in a vertex algebra was introduced by Frenkel and Zhu in the paper [15], it is a generalization of the coset construction considered by Kac-Peterson in representation theory ([22]) and Goddard-Kent-Olive in conformal field theory ([17]). Describing the generators (or even basis) of a commutant is generally a non-trivial problem (c.f. [10, 21]), the authors in the paper [23] reducing the problem of describing commutant in an appropriate category of vertex algebras to a question in commutative algebra, which is a new viewpoint, here we solve our problem by giving a characterization of our vertex operator algebra as a tensor product of a Heisenberg vertex operator algebra (with nonstandard conformal vector) and a Virasoro vertex operator algebra (constructed using new conformal vectors).

This paper is organized as follows: In section 2, we first review the definition of rank one twisted Heisenberg-Virasoro algebra $\mathcal{L}$ and define its restricted modules, then we prove that the category of restricted $\mathcal{L}$-modules is equivalent to the category of modules for a specific vertex algebra and we also present the equivalence between restricted $\mathcal{L}$-module category and ϕ-coordinated module category for certain vertex algebra. In section 3, we specifically study the structure theory of vertex operator algebra $V_{\mathcal{L}}(\ell_{123}, 0)$. In section 4, we study the relationship between the restricted module category of the rank two twisted Heisenberg-Virasoro Lie algebra $\mathcal{L}^*$ and ϕ-coordinated module category for a vertex algebra which is constructed based on a new Lie algebra.

2. Modules and ϕ-coordinated modules

In this section, we associate the rank one twisted Heisenberg-Virasoro algebra $\mathcal{L}$ with $V_{\mathcal{L}}(\ell_{123}, 0)$ in terms of vertex algebra with its module and ϕ-coordinated modules. More specifically, we show that there is a one-to-one correspondence between the restricted $\mathcal{L}$-modules of *level* ℓ_{123} and modules for the vertex algebra $V_{\mathcal{L}}(\ell_{123}, 0)$. And also the category of restricted $\mathcal{L}$-modules of level ℓ_{123} is equivalent to that of ϕ-coordinated modules for the vertex algebra $V_{\mathfrak{L}}(\ell_{123}, 0)$, where $\mathfrak{L}$ is a Lie algebra that is isomorphic to $\mathcal{L}$.

Throughout this paper, we denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{C}$, $\mathbb{C}^\times$ the set of nonnegative integers, integers, complex numbers, nonzero complex numbers respectively, and the symbols $x, x_1, x_2 \ldots$ denote mutually commuting independent formal variables. All vector spaces in this paper are considered to be over $\mathbb{C}$. For a vector space U, $U((x))$ is the vector space of lower truncated integral power series in x with coefficients in U, $U[[x]]$ is the vector space of nonnegative integral power series in x with coefficients in U, and $U[[x, x^{-1}]]$ is the vector space of doubly infinite integral power series in x with coefficients in U.

2.1. *Basic notions.* For later use, we know from [24] that

$$(x_1 - x_2)^m (\frac{\partial}{\partial x_2})^n x_1^{-1} \delta(\frac{x_2}{x_1}) = 0 \tag{2.1}$$

for $m > n$, $m, n \in \mathbb{N}$, where $\delta(\frac{x_1}{x_2}) = \sum\limits_{n \in \mathbb{Z}} x_1^n x_2^{-n}$.

For the definition of vertex (operator) algebra and its modules, we follow [24]. Let W be a general vector space. Set

$$\mathcal{E}(W) = \mathrm{Hom}(W, W((x))) \subset (\mathrm{EndW})[[x, x^{-1}]]. \tag{2.2}$$

The identity operator on W, denoted by $\mathbf{1}_W$, is a special element of $\mathcal{E}(W)$.

The following notion of locality was introduced in [27].

Definition 2.1. *Formal series $a(x), b(x) \in \mathcal{E}(W)$ are said to be* mutually local *if there exists a nonzero polynomial $(x_1 - x_2)^k$ with $k \in \mathbb{N}$ such that*

$$(x_1 - x_2)^k a(x_1) b(x_2) = (x_1 - x_2)^k b(x_2) a(x_1). \tag{2.3}$$

A subset (subspace) U of $\mathcal{E}(W)$ is said to be local *if any $a(x), b(x) \in U$ are mutually local.*

Recall the basic notions and results on ϕ-coordinated modules for vertex algebras ([28]). Set

$$\phi = \phi(x, z) = xe^z \in \mathbb{C}((x))[[z]],$$

which is fixed throughout the paper.

Definition 2.2. *Let V be a vertex algebra. A ϕ-*coordinated *V-*module *is a vector space W equipped with a linear map*

$$Y_W(\cdot, x) : V \longrightarrow \mathrm{Hom}(W, W((x))) \subset (\mathrm{EndW})[[x, x^{-1}]],$$

satisfying the conditions that $Y_W(\mathbf{1}, x) = \mathbf{1}_W$ and that for $u, v \in V$, there exists a nonzero polynomial $(x_1 - x_2)^k$ with $k \in \mathbb{N}$ such that

$$(x_1 - x_2)^k Y_W(u, x_1) Y_W(v, x_2) \in \mathrm{Hom}(W, W((x_1, x_2)))$$

and

$$(x_2 e^z - x_2)^k Y_W(Y(u, z)v, x_2) = ((x_1 - x_2)^k Y_W(u, x_1) Y_W(v, x_2))|_{x_1 = x_2 e^z}.$$

Let W be a general vector space, $a(x), b(x) \in \mathcal{E}(W)$. Assume that there exists a nonzero polynomial $p(x)$ such that

$$p(x,z)a(x)b(z) \in \mathrm{Hom}(W, W((x,z))). \tag{2.4}$$

Define $a(x)_n^e b(x) \in \mathcal{E}(W)$ for $n \in \mathbb{Z}$ in terms of generating function

$$Y_{\mathcal{E}}^e(a(x), z)b(x) = \sum_{n\in\mathbb{Z}} (a(x)_n^e b(x)) z^{-n-1}$$

by

$$Y_{\mathcal{E}}^e(a(x), z)b(x) = p(xe^z, x)^{-1}(p(x_1, x)a(x_1)b(x))|_{x_1 = xe^z},$$

where $p(x_1, x)$ is any nonzero polynomial such that (2.4) holds and $p(xe^z, x)^{-1}$ stands for the inverse of $p(xe^z, x)$ in $\mathbb{C}((x))((z))$. (Note that $p(xe^z, x)$ is a nonzero element in $\mathbb{C}((x))((z))$.) The definition of ϕ-coordinated module requires that $p(x,z)$ is of the form $(x-z)^k$ with $k \in \mathbb{N}$.

A subspace U of $\mathcal{E}(W)$ such that every ordered pair satisfies (2.4) is said to be $Y_{\mathcal{E}}^e$*-closed* if $a(x)_n^e b(x) \in U$ for $a(x), b(x) \in U$, $n \in \mathbb{Z}$. We denote by $\langle U\rangle_e$ the smallest $Y_{\mathcal{E}}^e$-closed subspace of $\mathcal{E}(W)$ that contains U and $\mathbf{1}_W$.

The following result was obtained in [28] (Theorem 5.4 and Proposition 5.3).

THEOREM 2.3. *Let U be a local subset of $\mathcal{E}(W)$. Then $(\langle U\rangle_e, Y_{\mathcal{E}}^e, \mathbf{1}_W)$ carries the structure of a vertex algebra and W is a ϕ-coordinated $\langle U\rangle_e$-module with $Y_W(a(x), z) = a(z)$ for $a(x) \in \langle U\rangle_e$.*

Now we are in a position to study the twisted Heisenberg-Virasoro algebra in terms of vertex algebra with its module and its ϕ-coordinated modules.

2.2. *Modules.* Firstly, we give the definition of the rank one twisted Heisenberg-Virasoro algebra $\mathcal{L}$ (see [2] or [5]).

DEFINITION 2.4. *The rank one twisted Heisenberg-Virasoro algebra $\mathcal{L}$ is a Lie algebra with the basis $\{L_n, I_n, c_1, c_2, c_3 | n \in \mathbb{Z}\}$, and the following Lie brackets:*

$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m+n,0}\frac{m^3-m}{12}c_1, \tag{2.5}$$

$$[L_m, I_n] = -nI_{m+n} - \delta_{m+n,0}(m^2+m)c_2, \tag{2.6}$$

$$[I_m, I_n] = m\delta_{m+n,0}c_3, \quad [\mathcal{L}, c_i] = 0, \ i = 1, 2, 3. \tag{2.7}$$

Clearly, $\mathrm{Span}\{L_n,\ c_1 \mid n \in \mathbb{Z}\}$ is a Virasoro algebra, $\mathrm{Span}\{I_n,\ c_3 \mid n \in \mathbb{Z}\backslash\{0\}\}$ is an infinite-dimensional Heisenberg algebra, we denote them by Vir, $\mathcal{H}$ respectively.

Form the generating functions as

$$L(x) = \sum_{n\in\mathbb{Z}} L_n x^{-n-2}, \quad I(x) = \sum_{n\in\mathbb{Z}} I_n x^{-n-1},$$

then the defining relations of $\mathcal{L}$ become to be

$$
\begin{aligned}
&[L(x_1), L(x_2)]\\
&\quad = \sum_{m,n\in\mathbb{Z}} (m-n)L_{m+n}x_1^{-m-2}x_2^{-n-2} + \sum_{m\in\mathbb{Z}} \frac{m^3-m}{12}c_1 x_1^{-m-2}x_2^{m-2}\\
&\quad = L^{'}(x_2)x_1^{-1}\delta\left(\frac{x_2}{x_1}\right) + 2L(x_2)\frac{\partial}{\partial x_2}x_1^{-1}\delta\left(\frac{x_2}{x_1}\right)\\
&\qquad + \frac{c_1}{12}\left(\frac{\partial}{\partial x_2}\right)^3 x_1^{-1}\delta\left(\frac{x_2}{x_1}\right),
\end{aligned} \tag{2.8}
$$

$$
\begin{aligned}
&[L(x_1), I(x_2)]\\
&\quad = -\sum_{m,n\in\mathbb{Z}} nI_{m+n}x_1^{-m-2}x_2^{-n-1} - \sum_{m\in\mathbb{Z}} (m^2+m)c_2 x_1^{-m-2}x_2^{m-1}\\
&\quad = I^{'}(x_2)x_1^{-1}\delta\left(\frac{x_2}{x_1}\right) + I(x_2)\frac{\partial}{\partial x_2}x_1^{-1}\delta\left(\frac{x_2}{x_1}\right)\\
&\qquad - \left(\frac{\partial}{\partial x_2}\right)^2 x_1^{-1}\delta\left(\frac{x_2}{x_1}\right)c_2,
\end{aligned} \tag{2.9}
$$

$$
[I(x_1), I(x_2)] = \sum_{m\in\mathbb{Z}} mc_3 x_1^{-m-1}x_2^{m-1} = \frac{\partial}{\partial x_2}x_1^{-1}\delta\left(\frac{x_2}{x_1}\right)c_3, \tag{2.10}
$$

where $L^{'}(x) = \frac{d}{dx}(L(x)), I^{'}(x) = \frac{d}{dx}(I(x))$.

We give the following definition.

Definition 2.5. *An $\mathcal{L}$-module W is said to be* restricted *if for any $w \in W, n \in \mathbb{Z}$, $L_n w = 0$ and $I_n w = 0$ for n sufficiently large, or equivalently, if $L(x), I(x) \in \mathcal{E}(W)$. We say an $\mathcal{L}$-module W is of* level ℓ_{123} *if the central element c_i acts as scalar ℓ_i for $i = 1, 2, 3$.*

Recall ([24]) that a Lie algebra $\mathfrak{g}$ equipped with a $\mathbb{Z}$-grading $\mathfrak{g} = \coprod_{n\in\mathbb{Z}} \mathfrak{g}_{(n)}$ is called a *$\mathbb{Z}$-graded Lie algebra* if

$$[\mathfrak{g}_{(m)}, \mathfrak{g}_{(n)}] \subset \mathfrak{g}_{(m+n)} \ \text{ for } m, n \in \mathbb{Z}.$$

A subalgebra $\mathfrak{h}$ of a $\mathbb{Z}$-graded Lie algebra $\mathfrak{g}$ is called a *graded subalgebra* if

$$\mathfrak{h} = \coprod_{n\in\mathbb{Z}} \mathfrak{h}_{(n)}, \text{ where } \mathfrak{h}_{(n)} = \mathfrak{h} \cap \mathfrak{g}_{(n)} \text{ for } n \in \mathbb{Z}.$$

In particular, $\mathfrak{g}_{(0)}$, $\mathfrak{g}_{(\pm)} = \coprod_{n\geq 1} \mathfrak{g}_{(\pm n)}$ and $\mathfrak{g}_{(0)} \oplus \mathfrak{g}_{(\pm)}$ are graded subalgebras.

For the twisted Heisenberg-Virasoro algebra $\mathcal{L}$, consider the following $\mathbb{Z}$-grading on $\mathcal{L}$

$$\mathcal{L} = \coprod_{n\in\mathbb{Z}} \mathcal{L}_{(n)}, \tag{2.11}$$

where

$$\mathcal{L}_{(0)} = \mathbb{C}L_0 \oplus \mathbb{C}I_0 \oplus \sum_{i=1}^{3} \mathbb{C}c_i, \ \text{ and } \ \mathcal{L}_{(n)} = \mathbb{C}L_{-n} \oplus \mathbb{C}I_{-n} \text{ for } n \neq 0.$$

It makes $\mathcal{L}$ a $\mathbb{Z}$-graded Lie algebra, and this grading is given by $\mathrm{ad}L_0$-eigenvalues. Then we have the graded subalgebras

$$\mathcal{L}_{(-)} = \coprod_{n\geq 1} \mathcal{L}_{(-n)} = \coprod_{n\geq 1} \mathbb{C}L_n \oplus \coprod_{n\geq 1} \mathbb{C}I_n,$$

$$\mathcal{L}_{(+)} = \coprod_{n\geq 1} \mathcal{L}_{(n)} = \coprod_{n\geq 1} \mathbb{C}L_{-n} \oplus \coprod_{n\geq 1} \mathbb{C}I_{-n},$$

$$\mathcal{L}_{(0)} \oplus \mathcal{L}_{(-)}, \ \text{ and } \ \mathcal{L}_{(0)} \oplus \mathcal{L}_{(+)}.$$

We also have the graded subalgebras

$$\mathcal{L}_{(\leq 1)} = \coprod_{n\leq 1} \mathbb{C}L_{-n} \oplus \coprod_{n\leq 0} \mathbb{C}I_{-n} \oplus \sum_{i=1}^{3} \mathbb{C}c_i, \tag{2.12}$$

$$\mathcal{L}_{(\geq 2)} = \coprod_{n\geq 2} \mathbb{C}L_{-n} \oplus \coprod_{n\geq 1} \mathbb{C}I_{-n}, \tag{2.13}$$

and the decomposition

$$\mathcal{L} = \mathcal{L}_{(\leq 1)} \oplus \mathcal{L}_{(\geq 2)}. \tag{2.14}$$

Let $\ell_i, i = 1, 2, 3$, be any complex numbers. Consider $\mathbb{C}$ as an $\mathcal{L}_{(\leq 1)}$-module with c_i acting as the scalar $\ell_i, i = 1, 2, 3$, and with $\coprod_{n\leq 1} \mathbb{C}L_{-n} \oplus \coprod_{n\leq 0} \mathbb{C}I_{-n}$ acting trivially. Denote this $\mathcal{L}_{(\leq 1)}$-module by $\mathbb{C}_{\ell_{123}}$. Form the induced module

$$V_{\mathcal{L}}(\ell_{123}, 0) = U(\mathcal{L}) \otimes_{U(\mathcal{L}_{(\leq 1)})} \mathbb{C}_{\ell_{123}}, \tag{2.15}$$

where $U(\cdot)$ denotes the universal enveloping algebra of a Lie algebra.

Set $\mathbf{1} = 1 \otimes 1 \in V_{\mathcal{L}}(\ell_{123}, 0)$. Define a linear operator $\overline{d}$ on $\mathcal{L}$ by

$$\overline{d}(c_i) = 0, \ \text{ for } i = 1, 2, 3,$$

$$\overline{d}(L_n) = -(n+1)L_{n-1}, \ \text{ and } \ \overline{d}(I_n) = -nI_{n-1}, \ \ n \in \mathbb{Z}.$$

It is easy to check that $\overline{d}$ is a derivation of the twisted Heisenberg-Virasoro algebra $\mathcal{L}$, so that $\overline{d}$ naturally extends to a derivation of the associative algebra $U(\mathcal{L})$. Clearly, $\overline{d}$ preserves the subspace $\coprod_{n\leq 1} \mathbb{C}L_{-n} \oplus \coprod_{n\leq 0} \mathbb{C}I_{-n} \oplus \sum_{i=1}^{3} \mathbb{C}(c_i - \ell_i)$ of $U(\mathcal{L})$. Since as a (left) $U(\mathcal{L})$-module,

$$V_{\mathcal{L}}(\ell_{123}, 0) \cong U(\mathcal{L})/U(\mathcal{L}) \left(\coprod_{n\leq 1} \mathbb{C}L_{-n} \oplus \coprod_{n\leq 0} \mathbb{C}I_{-n} \oplus \sum_{i=1}^{3} \mathbb{C}(c_i - \ell_i) \right),$$

it follows that $\overline{d}$ induces a linear operator on $V_{\mathcal{L}}(\ell_{123},0)$, which we denote by d. Then we have

$$d(\mathbf{1})=0, \text{ and } [d,L(x)]=\frac{d}{dx}L(x), \ [d,I(x)]=\frac{d}{dx}I(x).$$

From (2.8) to (2.10), using (2.1), we see that

$$(x_1-x_2)^4[L(x_1),L(x_2)]=0, \qquad (x_1-x_2)^3[L(x_1),I(x_2)]=0,$$

$$(x_1-x_2)^2[I(x_1),I(x_2)]=0.$$

By the Poincare-Birkhoff-Witt theorem, as a vector space we have

$$V_{\mathcal{L}}(\ell_{123},0)=U(\mathcal{L}_{(\geq 2)})\simeq S(\mathcal{L}_{(\geq 2)}).$$

And

$$V_{\mathcal{L}}(\ell_{123},0)=\coprod_{n\geq 0}V_{\mathcal{L}}(\ell_{123},0)_{(n)},$$

where $V_{\mathcal{L}}(\ell_{123},0)_{(0)}=\mathbb{C}_{\ell_{123}}$ and $V_{\mathcal{L}}(\ell_{123},0)_{(n)}$, $n\geq 1$, has a basis consisting of the vectors

$$I_{-k_1}\cdots I_{-k_s}L_{-m_1}\cdots L_{-m_r}\mathbf{1}$$

for $r,s\geq 0$, $m_1\geq\cdots\geq m_r\geq 2$, $k_1\geq\cdots\geq k_s\geq 1$ with $\sum_{i=1}^{r}m_i+\sum_{j=1}^{s}k_j=n$.

Then by Theorem 5.7.1 of [24] we get the following theorem.

THEOREM 2.6. *$V_{\mathcal{L}}(\ell_{123},0)$ is a vertex algebra, which is uniquely determined by the condition that $\mathbf{1}$ is the vacuum vector and*

$$Y(L_{-2}\mathbf{1},x)=L(x)\ \left(=\sum_{n\in\mathbb{Z}}L_nx^{-n-2}\right), \tag{2.16}$$

$$Y(I_{-1}\mathbf{1},x)=I(x)\ \left(=\sum_{n\in\mathbb{Z}}I_nx^{-n-1}\right). \tag{2.17}$$

The vertex operator map Y for this vertex algebra structure is given by

$$Y(I_{m_1}\cdots I_{m_s}L_{n_1}\cdots L_{n_r}\mathbf{1},x)=I(x)_{m_1}\cdots I(x)_{m_s}L(x)_{n_1+1}\cdots L(x)_{n_r+1}\mathbf{1}$$

for $r,s\geq 0$ and $n_1,\cdots,n_r,m_1,\cdots,m_s\in\mathbb{Z}$. Furthermore, $T=\{L_{-2}\mathbf{1},I_{-1}\mathbf{1}\}$ is the generating subset of $V_{\mathcal{L}}(\ell_{123},0)$.

CONVENTION: In our paper, for numbers of the form $n_1,\ldots,n_r$, we say $r\geq 0$, where $r=0$ means the element with subscript n_i's do not appear.

In the following, we denote by $\omega=L_{-2}\mathbf{1}$, $I=I_{-1}\mathbf{1}$, and $\omega^{'}=\frac{1}{2\ell_3}I_{-1}I_{-1}\mathbf{1}$ (for $\ell_3\neq 0$), note we have $\omega_n=L_{n-1}$, $(I)_n=(I_{-1}\mathbf{1})_n=I_n$ (this is why we denote by $I_{-1}\mathbf{1}$ the symbol I), for all $n\in\mathbb{Z}$.

REMARK 2.7. As a module for the twisted Heisenberg-Virasoro algebra, $V_{\mathcal{L}}(\ell_{123}, 0)$ is generated by $\mathbf{1}$ with the relations $c_i = \ell_i$ and $L_n\mathbf{1} = I_m\mathbf{1} = 0$ for $n \geq -1, m \geq 0$, $i = 1, 2, 3$. In fact, $V_{\mathcal{L}}(\ell_{123}, 0)$ is *universal* in the sense that for any module W of the twisted Heisenberg-Virasoro algebra $\mathcal{L}$ of level ℓ_{123} equipped with a vector $e \in W$ such that $L_n e = I_m e = 0$ for $n \geq -1, m \geq 0$, there exists a unique $\mathcal{L}$-module homomorphism from $V_{\mathcal{L}}(\ell_{123}, 0)$ to W sending $\mathbf{1}$ to e.

We now use Theorem 5.5.18 and Theorem 5.7.6 of [24] to prove the following result.

THEOREM 2.8. *Let W be any restricted module for the rank one twisted Heisenberg-Virasoro algebra $\mathcal{L}$ of level ℓ_{123}. Then there exists a unique module structure on W for $V_{\mathcal{L}}(\ell_{123}, 0)$ viewed as a vertex algebra such that*

$$Y_W(L_{-2}\mathbf{1}, x) = L(x) \ \left(= \sum_{n\in\mathbb{Z}} L_n x^{-n-2}\right), \tag{2.18}$$

$$Y_W(I_{-1}\mathbf{1}, x) = I(x) \ \left(= \sum_{n\in\mathbb{Z}} I_n x^{-n-1}\right). \tag{2.19}$$

The vertex operator map Y_W for this module structure is given by

$$Y_W(I_{m_1} \cdots I_{m_s} L_{n_1} \cdots L_{n_r}\mathbf{1}, x) = I(x)_{m_1} \cdots I(x)_{m_s} L(x)_{n_1+1} \cdots L(x)_{n_r+1}\mathbf{1}_W \tag{2.20}$$

for $r \geq 0, s \geq 0$ and $n_1, \cdots, n_r, m_1, \cdots, m_s \in \mathbb{Z}$.

PROOF. Set

$$U_W = \{L(x), I(x), \mathbf{1}_W\},$$

then U_W is a local subset of $\mathcal{E}(W)$. By Theorem 5.5.18 of [24], U_W generates a vertex algebra $\langle U_W\rangle$ with W a natural faithful module. Furthermore, $\langle U_W\rangle$ is the linear span of the elements of the form

$$a^{(1)}(x)_{n_1} \cdots a^{(r)}(x)_{n_r}\mathbf{1}_W$$

for $a^{(i)}(x) \in U_W$, $n_1, \ldots, n_r \in \mathbb{Z}$ with $r \geq 0$. $\langle U_W\rangle$ is an $\mathcal{L}$-module with L_n, I_n acting as $L(x)_{n+1}, I(x)_n$ for $n \in \mathbb{Z}$, so that $L(x)_n\mathbf{1}_W = 0$, $I(x)_n\mathbf{1}_W = 0$ for $n \geq 0$. In view of Remark 2.7, there exists a unique $\mathcal{L}$-module map ψ from $V_{\mathcal{L}}(\ell_{123}, 0)$ to $\langle U_W\rangle$ such that $\psi(\mathbf{1}) = \mathbf{1}_W$. Then

$$\psi(\omega_n v) = L(x)_n\psi(v) \ \text{ and } \psi(I_n v) = I(x)_n\psi(v), \ \text{ for } n \in \mathbb{Z}, v \in V_{\mathcal{L}}(\ell_{123}, 0).$$

The existence and uniqueness of $V_{\mathcal{L}}(\ell_{123}, 0)$-module structure on W now immediately follows from Theorem 5.7.6 of [24] with $T = \{\omega, I\}$. ∎

On the other hand, we have the following statement.

THEOREM 2.9. *Every module W for $V_{\mathcal{L}}(\ell_{123},0)$ viewed as a vertex algebra is naturally a restricted module for the rank one twisted Heisenberg-Virasoro algebra $\mathcal{L}$ of level ℓ_{123}, with $L(x)=Y_W(L_{-2}\mathbf{1},x)$, $I(x)=Y_W(I_{-1}\mathbf{1},x)$.*

PROOF. We have $(L_{-2}\mathbf{1})_i=\omega_i=L_{i-1}$, $(I_{-1}\mathbf{1})_i=I_i$ for $i\in\mathbb{Z}$. Then for $i\geq 0$,

$$\begin{aligned}(L_{-2}\mathbf{1})_iL_{-2}\mathbf{1}&=L_{i-1}L_{-2}\mathbf{1}=[L_{i-1},L_{-2}]\mathbf{1}\\&=(i+1)L_{i-3}\mathbf{1}+\delta_{i-3,0}\frac{(i-1)^3-(i-1)}{12}c_1\mathbf{1},\end{aligned}\tag{2.21}$$

$$\begin{aligned}(L_{-2}\mathbf{1})_iI_{-1}\mathbf{1}&=L_{i-1}I_{-1}\mathbf{1}=[L_{i-1},I_{-1}]\mathbf{1}\\&=I_{i-2}\mathbf{1}-\delta_{i-2,0}((i-1)^2+(i-1))c_2\mathbf{1},\end{aligned}\tag{2.22}$$

$$(I_{-1}\mathbf{1})_iI_{-1}\mathbf{1}=I_iI_{-1}\mathbf{1}=[I_i,I_{-1}]\mathbf{1}=i\delta_{i-1,0}c_3\mathbf{1}.\tag{2.23}$$

By Proposition 5.6.7 of [24], we get

$$\begin{aligned}&[Y_W(L_{-2}\mathbf{1},x_1),Y_W(L_{-2}\mathbf{1},x_2)]\\&=\sum_{i\geq 0}\frac{(-1)^i}{i!}Y_W((L_{-2}\mathbf{1})_iL_{-2}\mathbf{1},x_2)\left(\frac{\partial}{\partial x_1}\right)^i x_2^{-1}\delta\left(\frac{x_1}{x_2}\right)\\&=Y_W(L_{-3}\mathbf{1},x_2)x_2^{-1}\delta\left(\frac{x_1}{x_2}\right)\\&\quad-2Y_W(L_{-2}\mathbf{1},x_2)\left(\frac{\partial}{\partial x_1}\right)x_2^{-1}\delta\left(\frac{x_1}{x_2}\right)\\&\quad-\frac{1}{12}\left(\frac{\partial}{\partial x_1}\right)^3x_2^{-1}\delta\left(\frac{x_1}{x_2}\right)c_1\mathbf{1},\end{aligned}\tag{2.24}$$

$$\begin{aligned}&[Y_W(L_{-2}\mathbf{1},x_1),Y_W(I_{-1}\mathbf{1},x_2)]\\&=\sum_{i\geq 0}\frac{(-1)^i}{i!}Y_W((L_{-2}\mathbf{1})_iI_{-1}\mathbf{1},x_2)\left(\frac{\partial}{\partial x_1}\right)^i x_2^{-1}\delta\left(\frac{x_1}{x_2}\right)\\&=Y_W(I_{-2}\mathbf{1},x_2)x_2^{-1}\delta\left(\frac{x_1}{x_2}\right)\\&\quad-Y_W(I_{-1}\mathbf{1},x_2)\left(\frac{\partial}{\partial x_1}\right)x_2^{-1}\delta\left(\frac{x_1}{x_2}\right)-\left(\frac{\partial}{\partial x_1}\right)^2x_2^{-1}\delta\left(\frac{x_1}{x_2}\right)c_2\mathbf{1},\end{aligned}\tag{2.25}$$

$$\begin{aligned}&[Y_W(I_{-1}\mathbf{1},x_1),Y_W(I_{-1}\mathbf{1},x_2)]\\&=\sum_{i\geq 0}\frac{(-1)^i}{i!}Y_W((I_{-1}\mathbf{1})_iI_{-1}\mathbf{1},x_2)\left(\frac{\partial}{\partial x_1}\right)^i x_2^{-1}\delta\left(\frac{x_1}{x_2}\right)\\&=-\left(\frac{\partial}{\partial x_1}\right)x_2^{-1}\delta\left(\frac{x_1}{x_2}\right)c_3\mathbf{1}.\end{aligned}\tag{2.26}$$

Note that

$$Y_W(L_{-3}\mathbf{1},x)=Y_W(d(L_{-2}\mathbf{1}),x)=\frac{d}{dx}Y_W(L_{-2}\mathbf{1},x)$$

and

$$Y_W(I_{-2}\mathbf{1},x)=Y_W(d(I_{-1}\mathbf{1}),x)=\frac{d}{dx}Y_W(I_{-1}\mathbf{1},x).$$

With Proposition 2.3.6 of [24] and the fact

$$x_2^{-1}\delta\left(\frac{x_1}{x_2}\right)=x_1^{-1}\delta\left(\frac{x_1}{x_2}\right)=x_1^{-1}\delta\left(\frac{x_2}{x_1}\right),$$

we see that W is an $\mathcal{L}$-module of *level* ℓ_{123} with $L(x)=Y_W(L_{-2}\mathbf{1},x)$, and $I(x)=Y_W(I_{-1}\mathbf{1},x)$ for $L_{-2},I_{-1}\in\mathcal{L}$. Since W is a $V_{\mathcal{L}}(\ell_{123},0)$-module, by definition, $Y_W(L_{-2}\mathbf{1},x)$, $Y_W(I_{-1}\mathbf{1},x)\in\mathcal{E}(W)$. Therefore, W is a restricted $\mathcal{L}$-module of level ℓ_{123}. ∎

2.3. *ϕ-coordinated modules.* We now consider the case of ϕ-coordinated modules. Modifying the generating functions of $\mathcal{L}$ by a shift as follows:

$$\widetilde{L}(x)=\sum_{n\in\mathbb{Z}}L_nx^{-n},\quad \widetilde{I}(x)=\sum_{n\in\mathbb{Z}}I_nx^{-n},$$

then the defining relations of $\mathcal{L}$ become to be

$$\begin{aligned}
&[\widetilde{L}(x_1),\widetilde{L}(x_2)]\\
&\quad=\sum_{m,n\in\mathbb{Z}}(m-n)L_{m+n}x_1^{-m}x_2^{-n}+\sum_{m\in\mathbb{Z}}\frac{m^3-m}{12}c_1x_1^{-m}x_2^{m}\\
&\quad=\left(x_2\frac{\partial}{\partial x_2}\widetilde{L}(x_2)\right)\delta\left(\frac{x_2}{x_1}\right)+2\widetilde{L}(x_2)\left(x_2\frac{\partial}{\partial x_2}\right)\delta\left(\frac{x_2}{x_1}\right)\\
&\qquad+\frac{c_1}{12}\left(x_2\frac{\partial}{\partial x_2}\right)^3\delta\left(\frac{x_2}{x_1}\right)-\frac{c_1}{12}\left(x_2\frac{\partial}{\partial x_2}\right)\delta\left(\frac{x_2}{x_1}\right),
\end{aligned}\tag{2.27}$$

$$\begin{aligned}
&[\widetilde{L}(x_1),\widetilde{I}(x_2)]\\
&\quad=-\sum_{m,n\in\mathbb{Z}}nI_{m+n}x_1^{-m}x_2^{-n}-\sum_{m\in\mathbb{Z}}(m^2+m)c_2x_1^{-m}x_2^{m}\\
&\quad=\left(x_2\frac{\partial}{\partial x_2}\widetilde{I}(x_2)\right)\delta\left(\frac{x_2}{x_1}\right)+\widetilde{I}(x_2)\left(x_2\frac{\partial}{\partial x_2}\right)\delta\left(\frac{x_2}{x_1}\right)\\
&\qquad-\left(x_2\frac{\partial}{\partial x_2}\right)^2\delta\left(\frac{x_2}{x_1}\right)c_2-\left(x_2\frac{\partial}{\partial x_2}\right)\delta\left(\frac{x_2}{x_1}\right)c_2,
\end{aligned}\tag{2.28}$$

$$[\widetilde{I}(x_1),\widetilde{I}(x_2)]=\sum_{m\in\mathbb{Z}}mc_3x_1^{-m}x_2^{m}=\left(x_2\frac{\partial}{\partial x_2}\right)\delta\left(\frac{x_2}{x_1}\right)c_3.\tag{2.29}$$

We further set

$$\widehat{L}(x) = \widetilde{L}(x) - \frac{1}{24}c_1, \quad \widehat{I}(x) = \widetilde{I}(x) - c_2,$$

then we have

$$\begin{aligned}
&[\widehat{L}(x_1), \widehat{L}(x_2)] \\
&\quad = \left(x_2 \frac{\partial}{\partial x_2} \widehat{L}(x_2)\right) \delta\left(\frac{x_2}{x_1}\right) + 2\widehat{L}(x_2)\left(x_2 \frac{\partial}{\partial x_2}\right)\delta\left(\frac{x_2}{x_1}\right) \\
&\qquad + \frac{c_1}{12}\left(x_2 \frac{\partial}{\partial x_2}\right)^3 \delta\left(\frac{x_2}{x_1}\right),
\end{aligned} \tag{2.30}$$

$$\begin{aligned}
&[\widehat{L}(x_1), \widehat{I}(x_2)] \\
&\quad = \left(x_2 \frac{\partial}{\partial x_2} \widehat{I}(x_2)\right) \delta\left(\frac{x_2}{x_1}\right) + \widehat{I}(x_2)\left(x_2 \frac{\partial}{\partial x_2}\right)\delta\left(\frac{x_2}{x_1}\right) \\
&\qquad - \left(x_2 \frac{\partial}{\partial x_2}\right)^2 \delta\left(\frac{x_2}{x_1}\right) c_2,
\end{aligned} \tag{2.31}$$

$$[\widehat{I}(x_1), \widehat{I}(x_2)] = \left(x_2 \frac{\partial}{\partial x_2}\right) \delta\left(\frac{x_2}{x_1}\right) c_3. \tag{2.32}$$

Now we need a new Lie algebra to establish the connection between $\mathcal{L}$ and certain vertex algebra with respect to ϕ-coordinated modules.

Definition 2.10. *Let $\mathfrak{L}$ be a vector space spanned by the elements $\overline{L}_n, \overline{I}_n, c_i$, $n \in \mathbb{Z}, i = 1, 2, 3$, we define the brackets of $\mathfrak{L}$ as follows:*

$$[\overline{L}_m, \overline{L}_n] = (m-n)\overline{L}_{m+n-1} + \frac{m(m-1)(m-2)}{12}\delta_{m+n-2,0}c_1, \tag{2.33}$$

$$[\overline{L}_m, \overline{I}_n] = -n\overline{I}_{m+n-1} - (m^2 - m)\delta_{m+n-1,0}c_2, \tag{2.34}$$

$$[\overline{I}_m, \overline{I}_n] = m\delta_{m+n,0}c_3, \quad [\mathfrak{L}, c_i] = 0, \ \textit{for} \ i = 1, 2, 3. \tag{2.35}$$

It is straightforward to see that $\mathfrak{L}$ is a Lie algebra, and it is isomorphic to the rank one twisted Heisenberg-Virasoro algebra $\mathcal{L}$ via

$$\overline{L}_m \mapsto L_{m-1}, \ \ \overline{I}_m \mapsto I_m, \ \ c_i \mapsto c_i \ \text{ for } \ i = 1, 2, 3.$$

We set

$$\overline{L}(x) = \sum_{n \in \mathbb{Z}} \overline{L}_n x^{-n-1}, \qquad \overline{I}(x) = \sum_{n \in \mathbb{Z}} \overline{I}_n x^{-n-1},$$

then the definition relations of $\mathfrak{L}$ amount to

$$\begin{aligned}
&[\overline{L}(x_1), \overline{L}(x_2)] \\
&\quad = \sum_{m,n\in\mathbb{Z}} (m-n)\overline{L}_{m+n-1} x_1^{-m-1} x_2^{-n-1} \\
&\qquad + \sum_{m\in\mathbb{Z}} \frac{c_1}{12} m(m-1)(m-2) x_1^{-m-1} x_2^{m-3} \\
&\quad = \left(\frac{\partial}{\partial x_2}\overline{L}(x_2)\right) x_1^{-1}\delta\left(\frac{x_2}{x_1}\right) + 2\overline{L}(x_2)\frac{\partial}{\partial x_2} x_1^{-1}\delta\left(\frac{x_2}{x_1}\right) \\
&\qquad + \frac{c_1}{12}\left(\frac{\partial}{\partial x_2}\right)^3 x_1^{-1}\delta\left(\frac{x_2}{x_1}\right),
\end{aligned} \tag{2.36}$$

$$\begin{aligned}
&[\overline{L}(x_1), \overline{I}(x_2)] \\
&\quad = -\sum_{m,n\in\mathbb{Z}} n\overline{I}_{m+n-1} x_1^{-m-1} x_2^{-n-1} - \sum_{m\in\mathbb{Z}} (m^2-m) c_2 x_1^{-m-1} x_2^{m-2} \\
&\quad = \left(\frac{\partial}{\partial x_2}\overline{I}(x_2)\right) x_1^{-1}\delta\left(\frac{x_2}{x_1}\right) + \overline{I}(x_2)\frac{\partial}{\partial x_2} x_1^{-1}\delta\left(\frac{x_2}{x_1}\right) \\
&\qquad - \left(\frac{\partial}{\partial x_2}\right)^2 x_1^{-1}\delta\left(\frac{x_2}{x_1}\right) c_2,
\end{aligned} \tag{2.37}$$

$$[\overline{I}(x_1), \overline{I}(x_2)] = \sum_{m\in\mathbb{Z}} m c_3 x_1^{-m-1} x_2^{m-1} = \frac{\partial}{\partial x_2} x_1^{-1}\delta\left(\frac{x_2}{x_1}\right) c_3. \tag{2.38}$$

Set

$$\mathfrak{L}_{\geq 0} = \coprod_{n\geq 0} \overline{L}_n \oplus \coprod_{n\geq 0} \overline{I}_n \oplus \sum_{i=1}^{3} \mathbb{C}c_i, \qquad \mathfrak{L}_{<0} = \coprod_{n<0} \overline{L}_n \oplus \coprod_{n<0} \overline{I}_n.$$

Then $\mathfrak{L}_{\geq 0}$ and $\mathfrak{L}_{<0}$ are Lie subalgebras, and $\mathfrak{L} = \mathfrak{L}_{\geq 0} \oplus \mathfrak{L}_{<0}$ as a vector space. Let $\ell_i \in \mathbb{C}, i = 1, 2, 3$, we denote by $\mathbb{C}_{\ell_{123}} = \mathbb{C}$ the one-dimensional $\mathfrak{L}_{\geq 0}$-module with $\coprod_{n\geq 0} \overline{L}_n \oplus \coprod_{n\geq 0} \overline{I}_n$ acting trivially and c_i acting as ℓ_i for $i = 1, 2, 3$. Form the induced module

$$V_{\mathfrak{L}}(\ell_{123}, 0) = U(\mathfrak{L}) \otimes_{U(\mathfrak{L}_{\geq 0})} \mathbb{C}_{\ell_{123}}.$$

Set $\mathbf{1} = 1 \otimes 1 \in V_{\mathfrak{L}}(\ell_{123}, 0)$. Similarly, one can show that there exists a natural vertex algebra structure on $V_{\mathfrak{L}}(\ell_{123}, 0)$ with a linear operator $\overline{d}$ on $\mathfrak{L}$ defined by

$$\overline{d}(c_i) = 0, \ \text{ for } \ i = 1, 2, 3,$$
$$\overline{d}(\overline{L}_n) = -n\overline{L}_{n-1}, \ \ \overline{d}(\overline{I}_n) = -n\overline{I}_{n-1}, \ \ n \in \mathbb{Z},$$

and it is uniquely determined by the condition that $\mathbf{1}$ is the vacuum vector,

$$Y(\overline{L}_{-1}\mathbf{1}, x) = \overline{L}(x) \left(= \sum_{n\in\mathbb{Z}} \overline{L}_n x^{-n-1} \right), \tag{2.39}$$

$$Y(\overline{I}_{-1}\mathbf{1},x)=\overline{I}(x)\left(=\sum_{n\in\mathbb{Z}}\overline{I}_nx^{-n-1}\right). \tag{2.40}$$

The vertex operator map Y for this vertex algebra structure is given by

$$Y(\overline{I}_{m_1}\cdots\overline{I}_{m_s}\overline{L}_{n_1}\cdots\overline{L}_{n_r}\mathbf{1},x)=\overline{I}(x)_{m_1}\cdots\overline{I}(x)_{m_s}\overline{L}(x)_{n_1}\cdots\overline{L}(x)_{n_r}\mathbf{1}$$

for $r\geq 0, s\geq 0$ and $n_1,\cdots,n_r,m_1,\cdots,m_s\in\mathbb{Z}$. Furthermore, $T=\{\overline{L}_{-1}\mathbf{1},\overline{I}_{-1}\mathbf{1}\}$ is a generating subset of $V_{\mathfrak{L}}(\ell_{123},0)$.

Remark 2.11. As a module for the new Lie algebra $\mathfrak{L}$, $V_{\mathfrak{L}}(\ell_{123},0)$ is generated by $\mathbf{1}$ with the relations $c_i=\ell_i$ and $\overline{L}_n\mathbf{1}=\overline{I}_n\mathbf{1}=0$ for $n\geq 0$, $i=1,2,3$. $V_{\mathfrak{L}}(\ell_{123},0)$ is *universal* in the sense that for any module W of $\mathfrak{L}$ of level ℓ_{123} equipped with a vector $e\in W$ such that $\overline{L}_ne=\overline{I}_ne=0$ for $n\geq 0$, there exists a unique $\mathfrak{L}$-module homomorphism from $V_{\mathfrak{L}}(\ell_{123},0)$ to W sending $\mathbf{1}$ to e.

And we have the following result.

Theorem 2.12. *Let W be a restricted $\mathcal{L}$-module of level ℓ_{123}. Then there exists a ϕ-coordinated $V_{\mathfrak{L}}(\ell_{123},0)$-module structure $Y_W(\cdot,x)$ on W, which is uniquely determined by*

$$Y_W(\overline{L}_{-1}\mathbf{1},x)=\widehat{L}(x)\ \textit{ and }\ Y_W(\overline{I}_{-1}\mathbf{1},x)=\widehat{I}(x)\quad \textit{ for } \overline{L}_{-1},\overline{I}_{-1}\in\mathfrak{L}.$$

The vertex operator map Y_W for this module structure is given by

$$Y_W(\overline{I}_{m_1}\cdots\overline{I}_{m_s}\overline{L}_{n_1}\cdots\overline{L}_{n_r}\mathbf{1},x)=\widehat{I}(x)^e_{m_1}\cdots\widehat{I}(x)^e_{m_s}\widehat{L}(x)^e_{n_1}\cdots\widehat{L}(x)^e_{n_r}\mathbf{1}_W$$

for $r\geq 0, s\geq 0$ and $n_1,\cdots,n_r,m_1,\cdots,m_s\in\mathbb{Z}$.

Proof. Since $T=\{\overline{L}_{-1}\mathbf{1},\overline{I}_{-1}\mathbf{1}\}$ generates $V_{\mathfrak{L}}(\ell_{123},0)$ as a vertex algebra, the uniqueness is clear. We now prove the existence. Set $U_W=\{\mathbf{1}_W\}\cup\{\widehat{L}(x),\widehat{I}(x)\}\subset\mathcal{E}(W)$. From (2.30) to (2.32), by using Lemma 2.1 of [29], we see that

$$(x_1-x_2)^4[\widehat{L}(x_1),\widehat{L}(x_2)]=0,\ \ (x_1-x_2)^3[\widehat{L}(x_1),\widehat{I}(x_2)]=0,$$
$$(x_1-x_2)^2[\widehat{I}(x_1),\widehat{I}(x_2)]=0.$$

Then U_W is a local subset of $\mathcal{E}(W)$, Theorem 2.3 tells us U_W generates a vertex algebra $\langle U_W\rangle_e$ under the vertex operator operation $Y^e_{\mathcal{E}}$ with W a ϕ-coordinated module, and $Y_W(a(x),z)=a(z)$ for $a(x)\in\langle U_W\rangle_e$. Using Lemma 4.13 or Proposition 4.14 of [29], together with (2.30), (2.31) and (2.32), we have

$$\widehat{L}(x)^e_i\widehat{L}(x)=0\ \text{ for } i=2 \text{ and } i\geq 4,$$
$$\widehat{L}(x)^e_3\widehat{L}(x)=\frac{\ell_1}{2}\mathbf{1}_W,$$
$$\widehat{L}(x)^e_1\widehat{L}(x)=2\widehat{L}(x),$$

$$\begin{gathered}
\widehat{L}(x)_0^e\widehat{L}(x)=x\frac{\partial}{\partial x}\widehat{L}(x),\\
\widehat{L}(x)_i^e\widehat{I}(x)=0\ \text{ for } i\geq 3,\\
\widehat{L}(x)_2^e\widehat{I}(x)=-2\ell_2\mathbf{1}_W,\\
\widehat{L}(x)_1^e\widehat{I}(x)=\widehat{I}(x),\\
\widehat{L}(x)_0^e\widehat{I}(x)=x\frac{\partial}{\partial x}\widehat{I}(x),
\end{gathered}$$

and

$$\begin{gathered}
\widehat{I}(x)_i^e\widehat{I}(x)=0\ \text{ for } i=0 \text{ and } i\geq 2,\\
\widehat{I}(x)_1^e\widehat{I}(x)=\ell_3\mathbf{1}_W.
\end{gathered}$$

Then by Borcherds' commutator formula we have

$$\begin{aligned}
&[Y_{\mathcal{E}}^e(\widehat{L}(x),x_1),Y_{\mathcal{E}}^e(\widehat{L}(x),x_2)]\\
&\qquad=\sum_{i\geq 0}Y_{\mathcal{E}}^e(\widehat{L}(x)_i^e\widehat{L}(x),x_2)\frac{1}{i!}\left(\frac{\partial}{\partial x_2}\right)^i x_1^{-1}\delta\left(\frac{x_2}{x_1}\right)\\
&\qquad=Y_{\mathcal{E}}^e(x\frac{\partial}{\partial x}\widehat{L}(x),x_2)x_1^{-1}\delta\left(\frac{x_2}{x_1}\right)+2Y_{\mathcal{E}}^e(\widehat{L}(x),x_2)\frac{\partial}{\partial x_2}x_1^{-1}\delta\left(\frac{x_2}{x_1}\right)\\
&\qquad\quad+\frac{\ell_1}{12}\mathbf{1}_W\left(\frac{\partial}{\partial x_2}\right)^3 x_1^{-1}\delta\left(\frac{x_2}{x_1}\right),\\
&[Y_{\mathcal{E}}^e(\widehat{L}(x),x_1),Y_{\mathcal{E}}^e(\widehat{I}(x),x_2)]\\
&\qquad=\sum_{i\geq 0}Y_{\mathcal{E}}^e(\widehat{L}(x)_i^e\widehat{I}(x),x_2)\frac{1}{i!}\left(\frac{\partial}{\partial x_2}\right)^i x_1^{-1}\delta\left(\frac{x_2}{x_1}\right)\\
&\qquad=Y_{\mathcal{E}}^e(x\frac{\partial}{\partial x}\widehat{I}(x),x_2)x_1^{-1}\delta\left(\frac{x_2}{x_1}\right)+Y_{\mathcal{E}}^e(\widehat{I}(x),x_2)\frac{\partial}{\partial x_2}x_1^{-1}\delta\left(\frac{x_2}{x_1}\right)\\
&\qquad\quad-\ell_2\mathbf{1}_W\left(\frac{\partial}{\partial x_2}\right)^2 x_1^{-1}\delta\left(\frac{x_2}{x_1}\right),
\end{aligned}$$

and

$$\begin{aligned}
&[Y_{\mathcal{E}}^e(\widehat{I}(x),x_1),Y_{\mathcal{E}}^e(\widehat{I}(x),x_2)]\\
&\qquad=\sum_{i\geq 0}Y_{\mathcal{E}}^e(\widehat{I}(x)_i^e\widehat{I}(x),x_2)\frac{1}{i!}\left(\frac{\partial}{\partial x_2}\right)^i x_1^{-1}\delta\left(\frac{x_2}{x_1}\right)\\
&\qquad=\ell_3\mathbf{1}_W\frac{\partial}{\partial x_2}x_1^{-1}\delta\left(\frac{x_2}{x_1}\right).
\end{aligned}$$

Comparing these with (2.36) to (2.38), we see that $\langle U_W\rangle_e$ is an $\mathfrak{L}$-module of *level* ℓ_{123} with $\overline{L}(z),\overline{I}(z)$ acting as $Y_{\mathcal{E}}^e(\widehat{L}(x),z),Y_{\mathcal{E}}^e(\widehat{I}(x),z)$ respectively, and $\frac{\partial}{\partial z}\overline{L}(z),\frac{\partial}{\partial z}\overline{I}(z)$ acting as $Y_{\mathcal{E}}^e(x\frac{\partial}{\partial x}\widehat{L}(x),z)$, $Y_{\mathcal{E}}^e(x\frac{\partial}{\partial x}\widehat{I}(x),z)$ respectively. Since $\widehat{L}(x)_n^e\mathbf{1}_W=\widehat{I}(x)_n^e\mathbf{1}_W=0$ for $n\neq -1$, and $\widehat{L}(x)_{-1}^e\mathbf{1}_W=\widehat{L}(x)$, $\widehat{I}(x)_{-1}^e\mathbf{1}_W=$

$\widehat{I}(x)$, we have $\overline{L}_n\mathbf{1}_W=\overline{I}_n\mathbf{1}_W=0$ for $n\geq 0$. Then it follows from Remark 2.11 that there exists a unique $\mathfrak{L}$-module homomorphism ψ from $V_{\mathfrak{L}}(\ell_{123},0)$ to $\langle U_W\rangle_e$ with $\psi(\mathbf{1})=\mathbf{1}_W$. So

$$\psi(\overline{L}_{-1}\mathbf{1})=\widehat{L}(x)^e_{-1}\mathbf{1}_W=\widehat{L}(x)\in\langle U_W\rangle_e,$$

$$\psi(\overline{I}_{-1}\mathbf{1})=\widehat{I}(x)^e_{-1}\mathbf{1}_W=\widehat{I}(x)\in\langle U_W\rangle_e.$$

Now for $s\in\mathbb{Z}$, $v\in V_{\mathfrak{L}}(\ell_{123},0)$, with (2.39), (2.40) we have

$$\psi((\overline{L}_{-1}\mathbf{1})_s v)=\psi(\overline{L}_s v)=\widehat{L}(x)^e_s\psi(v)=\psi(\overline{L}_{-1}\mathbf{1})^e_s\psi(v),$$

$$\psi((\overline{I}_{-1}\mathbf{1})_s v)=\psi(\overline{I}_s v)=\widehat{I}(x)^e_s\psi(v)=\psi(\overline{I}_{-1}\mathbf{1})^e_s\psi(v),$$

Since $T=\{\overline{L}_{-1}\mathbf{1},\overline{I}_{-1}\mathbf{1}\}$ generates $V_{\mathfrak{L}}(\ell_{123},0)$ as a vertex algebra, it follows from Proposition 5.7.9 of [24] that ψ is a homomorphism of vertex algebra. And then W becomes a ϕ-coordinated module of $V_{\mathfrak{L}}(\ell_{123},0)$ with

$$Y_W(\overline{L}_{-1}\mathbf{1},x)=\widehat{L}(x),\quad Y_W(\overline{I}_{-1}\mathbf{1},x)=\widehat{I}(x)\quad\text{for }\overline{L}_{-1},\overline{I}_{-1}\in\mathfrak{L},$$

and

$$Y_W(\overline{I}_{m_1}\cdots\overline{I}_{m_s}\overline{L}_{n_1}\cdots\overline{L}_{n_r}\mathbf{1},x)=\widehat{I}(x)^e_{m_1}\cdots\widehat{I}(x)^e_{m_s}\widehat{L}(x)^e_{n_1}\cdots\widehat{L}(x)^e_{n_r}\mathbf{1}_W$$

for $r\geq 0, s\geq 0$ and $n_1,\cdots,n_r,m_1,\cdots,m_s\in\mathbb{Z}$. □

On the other hand, we have the following statement.

THEOREM 2.13. *Let W be a ϕ-coordinated $V_{\mathfrak{L}}(\ell_{123},0)$-module. Then W is a restricted $\mathcal{L}$-module of level ℓ_{123} with $\widehat{L}(x)=Y_W(\overline{L}_{-1}\mathbf{1},x)$, and $\widehat{I}(x)=Y_W(\overline{I}_{-1}\mathbf{1},x)$ for $\overline{L}_{-1},\overline{I}_{-1}\in\mathfrak{L}$.*

PROOF. For $\overline{L}_{-1},\overline{I}_{-1}\in\mathfrak{L}$, since $Y(\overline{L}_{-1}\mathbf{1},x)=\overline{L}(x)$, $Y(\overline{I}_{-1}\mathbf{1},x)=\overline{I}(x)$, from the identities (2.36) to (2.38), by using (2.1) we see that

$$(x_1-x_2)^4[Y(\overline{L}_{-1}\mathbf{1},x_1),Y(\overline{L}_{-1}\mathbf{1},x_2)]=0,$$

$$(x_1-x_2)^3[Y(\overline{L}_{-1}\mathbf{1},x_1),Y(\overline{I}_{-1}\mathbf{1},x_2)]=0,$$

$$(x_1-x_2)^2[Y(\overline{I}_{-1}\mathbf{1},x_1),Y(\overline{I}_{-1}\mathbf{1},x_2)]=0.$$

Note that for $i\geq 0$, we have

$$\begin{aligned}(\overline{L}_{-1}\mathbf{1})_i\overline{L}_{-1}\mathbf{1}&=\overline{L}_i\overline{L}_{-1}\mathbf{1}=[\overline{L}_i,\overline{L}_{-1}]\mathbf{1}\\&=(i+1)\overline{L}_{i-2}\mathbf{1}+\frac{i(i-1)(i-2)}{12}\delta_{i-3,0}\ell_1\mathbf{1},\\(\overline{L}_{-1}\mathbf{1})_i\overline{I}_{-1}\mathbf{1}&=\overline{L}_i\overline{I}_{-1}\mathbf{1}=[\overline{L}_i,\overline{I}_{-1}]\mathbf{1}=\overline{I}_{i-2}\mathbf{1}-(i^2-i)\delta_{i-2,0}\ell_2\mathbf{1},\end{aligned}$$

and

$$(\overline{I}_{-1}\mathbf{1})_i\overline{I}_{-1}\mathbf{1}=\overline{I}_i\overline{I}_{-1}\mathbf{1}=[\overline{I}_i,\overline{I}_{-1}]\mathbf{1}=i\delta_{i-1,0}\ell_3\mathbf{1}.$$

By Proposition 5.9 of [28], we have

$$\begin{aligned}
&[Y_W(\overline{L}_{-1}\mathbf{1},x_1),Y_W(\overline{L}_{-1}\mathbf{1},x_2)]\\
&\qquad=\mathrm{Res}_{x_0}x_1^{-1}\delta\left(\frac{x_2e^{x_0}}{x_1}\right)x_2e^{x_0}Y_W(Y(\overline{L}_{-1}\mathbf{1},x_0)\overline{L}_{-1}\mathbf{1},x_2)\\
&\qquad=Y_W(\overline{L}_{-2}\mathbf{1},x_2)\delta\left(\frac{x_2}{x_1}\right)+2Y_W(\overline{L}_{-1}\mathbf{1},x_2)\left(x_2\frac{\partial}{\partial x_2}\right)\delta\left(\frac{x_2}{x_1}\right)\\
&\qquad\quad+\frac{\ell_1\mathbf{1}_W}{12}\left(x_2\frac{\partial}{\partial x_2}\right)^3\delta\left(\frac{x_2}{x_1}\right),\\
&[Y_W(\overline{L}_{-1}\mathbf{1},x_1),Y_W(\overline{I}_{-1}\mathbf{1},x_2)]\\
&\qquad=\mathrm{Res}_{x_0}x_1^{-1}\delta\left(\frac{x_2e^{x_0}}{x_1}\right)x_2e^{x_0}Y_W(Y(\overline{L}_{-1}\mathbf{1},x_0)\overline{I}_{-1}\mathbf{1},x_2)\\
&\qquad=Y_W(\overline{I}_{-2}\mathbf{1},x_2)\delta\left(\frac{x_2}{x_1}\right)+Y_W(\overline{I}_{-1}\mathbf{1},x_2)\left(x_2\frac{\partial}{\partial x_2}\right)\delta\left(\frac{x_2}{x_1}\right)\\
&\qquad\quad-\left(x_2\frac{\partial}{\partial x_2}\right)^2\delta\left(\frac{x_2}{x_1}\right)\ell_2\mathbf{1}_W,
\end{aligned}$$

and

$$\begin{aligned}
&[Y_W(\overline{I}_{-1}\mathbf{1},x_1),Y_W(\overline{I}_{-1}\mathbf{1},x_2)]\\
&\qquad=\mathrm{Res}_{x_0}x_1^{-1}\delta\left(\frac{x_2e^{x_0}}{x_1}\right)x_2e^{x_0}Y_W(Y(\overline{I}_{-1}\mathbf{1},x_0)\overline{I}_{-1}\mathbf{1},x_2)\\
&\qquad=\left(x_2\frac{\partial}{\partial x_2}\right)\delta\left(\frac{x_2}{x_1}\right)\ell_3\mathbf{1}_W.
\end{aligned}$$

For a ϕ-coordinated module, by Lemma 3.7 of [28], we have

$$Y_W(\overline{L}_{-2}\mathbf{1},x)=Y_W(d(\overline{L}_{-1}\mathbf{1}),x)=x\frac{\partial}{\partial x}Y_W(\overline{L}_{-1}\mathbf{1},x)$$

and

$$Y_W(\overline{I}_{-2}\mathbf{1},x)=Y_W(d(\overline{I}_{-1}\mathbf{1}),x)=x\frac{\partial}{\partial x}Y_W(\overline{I}_{-1}\mathbf{1},x).$$

Then W is an $\mathcal{L}$-module of *level* ℓ_{123} with $\widehat{L}(x)=Y_W(\overline{L}_{-1}\mathbf{1},x)$, and $\widehat{I}(x)=Y_W(\overline{I}_{-1}\mathbf{1},x)$ for $\overline{L}_{-1},\overline{I}_{-1}\in\mathfrak{L}$. Since W is a ϕ-coordinated $V_{\mathfrak{L}}(\ell_{123},0)$-module, by definition, $Y_W(\overline{L}_{-1}\mathbf{1},x)$, $Y_W(\overline{I}_{-1}\mathbf{1},x)\in\mathcal{E}(W)$. Therefore, W is a restricted $\mathcal{L}$-module of level ℓ_{123}. ∎

3. Structures of twisted Heisenberg-Virasoro vertex operator algebra $V_{\mathcal{L}}(\ell_{123},0)$ and its irreducible modules

In this section, we first show that $V_{\mathcal{L}}(\ell_{123},0)$ is a vertex operator algebra and characterize its irreducible modules. Then we study the structure theory of $V_{\mathcal{L}}(\ell_{123},0)$ and get the corresponding results of the simple vertex operator algebra coming from it. Specifically, we study its Zhu's algebra, rationality, C_2-cofiniteness, regularity, unitarity and the commutant of Heisenberg vertex operator subalgebra. In the process, we get the result that $V_{\mathcal{L}}(\ell_{123}, 0)$ can be characterized as a tensor product vertex operator algebra.

3.1. *Vertex operator algebra $V_{\mathcal{L}}(\ell_{123}, 0)$ and its irreducible modules.* For a $\mathbb{Z}$-graded Lie algebra $\mathfrak{g} = \coprod_{n\in\mathbb{Z}} \mathfrak{g}_{(n)}$, a $\mathbb{C}$-*graded* $\mathfrak{g}$-*module* is a $\mathfrak{g}$-module W equipped with a $\mathbb{C}$-grading $W = \coprod_{r\in\mathbb{C}} W_{(r)}$ such that

$$\mathfrak{g}_{(n)} W_{(r)} \subset W_{(n+r)} \text{ for } n \in \mathbb{Z},\ r \in \mathbb{C}. \tag{3.1}$$

From the above section, we know that $V_{\mathcal{L}}(\ell_{123}, 0)$ is $\mathbb{Z}$-graded by L_0-eigenvalues, and clearly, the two grading restriction conditions in the definition of vertex operator algebra hold for $V_{\mathcal{L}}(\ell_{123}, 0)$. It is also clear that $V_{\mathcal{L}}(\ell_{123}, 0)$ is a restricted $\mathcal{L}$-module, a restricted Vir-module (and also a restricted $\mathcal{H}$-module). In order to say that it has a vertex operator algebra structure, it remains to check (5.7.22) or (5.7.23) of Theorem 5.7.4 of [24] which is straightforward, so we have the following result.

Proposition 3.1. $(V_{\mathcal{L}}(\ell_{123}, 0), Y, \mathbf{1}, L_{-2}\mathbf{1})$ *is a vertex operator algebra with conformal vector $\omega = L_{-2}\mathbf{1}$ and of central charge ℓ_1.*

Remark 3.2. $V_{\mathcal{L}}(\ell_{123}, 0)$ is generated by the conformal vector $\omega = L_{-2}\mathbf{1}$ and $I = I_{-1}\mathbf{1}$. It is not a minimal vertex operator algebra, for example, it has a proper vertex operator subalgebra $V_{Vir}(\ell_1, 0)$ (with the same conformal vector ω).

We now investigate the modules of $V_{\mathcal{L}}(\ell_{123}, 0)$ viewed as a vertex operator algebra. By Theorem 2.8, if further W is $\mathbb{C}$-graded by L_0-eigenvalues, then W is a module for $V_{\mathcal{L}}(\ell_{123}, 0)$ viewed as a vertex operator algebra, possibly without the two grading restrictions.

By Theorem 2.8 and Theorem 2.9 we have the following statement.

Theorem 3.3. *The modules for $V_{\mathcal{L}}(\ell_{123}, 0)$ viewed as a vertex operator algebra (i.e. $\mathbb{C}$-graded by L_0-eigenvalues and with the two grading restrictions) are exactly those restricted modules for the Lie algebra $\mathcal{L}$ of level ℓ_{123} that are $\mathbb{C}$-graded by L_0-eigenvalues and with the two grading restrictions. Furthermore, for any $V_{\mathcal{L}}(\ell_{123}, 0)$-module W, the $V_{\mathcal{L}}(\ell_{123}, 0)$-submodules of W are exactly the submodules of W for $\mathcal{L}$, and these submodules are in particular graded.*

Next, we will modify the construction of the $\mathcal{L}$-module $V_{\mathcal{L}}(\ell_{123}, 0)$ to get a certain natural family of restricted $\mathcal{L}$-modules of level ℓ_{123} that are $\mathbb{C}$-graded by L_0-eigenvalues and satisfy the two grading restrictions. Note such $\mathcal{L}$-modules are naturally modules for the vertex operator algebra $V_{\mathcal{L}}(\ell_{123}, 0)$ by the above theorem.

Consider $\mathbb{C}$ as an $\mathcal{L}_{(0)}$-module with c_i acting as the scalar ℓ_i, $i = 1, 2, 3$, L_0 acting as h_1 and I_0 acting as h_2, where $\ell_1, \ell_2, \ell_3, h_1, h_2 \in \mathbb{C}$.

Let $\mathcal{L}_{(-)}$ acting trivially on $\mathbb{C}$, making $\mathbb{C}$ an $(\mathcal{L}_{(-)} \oplus \mathcal{L}_{(0)})$-module, which we denote by $\mathbb{C}_{\ell_{123},h_{12}}$. Form the induced module

$$M_{\mathcal{L}}(\ell_{123}, h_1, h_2) = U(\mathcal{L}) \otimes_{U(\mathcal{L}_{(-)} \oplus \mathcal{L}_{(0)})} \mathbb{C}_{\ell_{123},h_{12}}. \tag{3.2}$$

Again from the Poincare-Birkhoff-Witt theorem, as a vector space we have

$$M_{\mathcal{L}}(\ell_{123}, h_1, h_2) = U(\mathcal{L}_{(+)}) \otimes \mathbb{C}_{\ell_{123},h_{12}} = U(\mathcal{L}_{(+)}) \simeq S(\mathcal{L}_{(+)}). \tag{3.3}$$

We naturally consider $\mathbb{C}_{\ell_{123},h_{12}}$ as a subspace of $M_{\mathcal{L}}(\ell_{123}, h_1, h_2)$ and set

$$\mathbf{1}_{(\ell_{123},h_{12})} = 1 \in \mathbb{C}_{\ell_{123},h_{12}} \subset M_{\mathcal{L}}(\ell_{123}, h_1, h_2).$$

Then

$$M_{\mathcal{L}}(\ell_{123}, h_1, h_2) = \coprod_{n \geq 0} M_{\mathcal{L}}(\ell_{123}, h_1, h_2)_{(n+h_1)},$$

where $M_{\mathcal{L}}(\ell_{123}, h_1, h_2)_{(h_1)} = \mathbb{C}_{\ell_{123},h_{12}}$ and $M_{\mathcal{L}}(\ell_{123}, h_1, h_2)_{(n+h_1)}$ for $n \geq 1$ is the L_0-eigenspace of eigenvalue $n + h_1$. $M_{\mathcal{L}}(\ell_{123}, h_1, h_2)_{(n+h_1)}$ has a basis consisting of the vectors

$$I_{-k_1} \cdots I_{-k_s} L_{-m_1} \cdots L_{-m_r} \mathbf{1}_{(\ell_{123},h_{12})}$$

for r, $s \geq 0$, $m_1 \geq \cdots \geq m_r \geq 1$, $k_1 \geq \cdots \geq k_s \geq 1$ with $\sum_{i=1}^{r} m_i + \sum_{j=1}^{s} k_j = n$. Hence, as a module for $\mathcal{L}$ of level ℓ_{123}, $M_{\mathcal{L}}(\ell_{123}, h_1, h_2)$ is $\mathbb{C}$-graded by L_0-eigenvalues.

Consequently, $M_{\mathcal{L}}(\ell_{123}, h_1, h_2)$ with the given $\mathbb{C}$-grading satisfies the grading restriction conditions. This in particular implies that $M_{\mathcal{L}}(\ell_{123}, h_1, h_2)$ is a restricted $\mathcal{L}$-module.

Thus from Theorem 3.3 we immediately have the following theorem.

THEOREM 3.4. *For any complex numbers $\ell_1, \ell_2, \ell_3, h_1$ and h_2, $W = M_{\mathcal{L}}(\ell_{123}, h_1, h_2)$ has a unique module structure for the vertex operator algebra $V_{\mathcal{L}}(\ell_{123}, 0)$ such that $Y_W(\omega, x) = L(x)$ and $Y_W(I_{-1}\mathbf{1}, x) = I(x)$.*

REMARK 3.5. The $\mathcal{L}$-module $M_{\mathcal{L}}(\ell_{123}, h_1, h_2)$ is commonly referred to in the literature as the *Verma module* in the papers [2, 5]. As a module for $\mathcal{L}$, $M_{\mathcal{L}}(\ell_{123}, h_1, h_2)$ is generated by $\mathbf{1}_{(\ell_{123},h_{12})}$ with the relations

$$L_0 \mathbf{1}_{(\ell_{123},h_{12})} = h_1 \mathbf{1}_{(\ell_{123},h_{12})},\ I_0 \mathbf{1}_{(\ell_{123},h_{12})} = h_2 \mathbf{1}_{(\ell_{123},h_{12})},\ c_i = \ell_i, i = 1, 2, 3,$$

$$\text{and}\ \ L_n \mathbf{1}_{(\ell_{123},h_{12})} = 0,\ I_n \mathbf{1}_{(\ell_{123},h_{12})} = 0 \text{ for } n \geq 1.$$

$M_{\mathcal{L}}(\ell_{123}, h_1, h_2)$ is *universal* in the sense that for any $\mathcal{L}$-module W of level ℓ_{123} equipped with a vector v such that $L_0 v = h_1 v, I_0 v = h_2 v, L_n v = 0, I_n v = 0$ for all $n \geq 1$, there exists a unique module map $M_{\mathcal{L}}(\ell_{123}, h_1, h_2) \longrightarrow W$ sending $\mathbf{1}_{(\ell_{123},h_{12})}$ to v.

In general, $M_{\mathcal{L}}(\ell_{123}, h_1, h_2)$ as a module for $\mathcal{L}$ may be reducible, then it is a reducible $V_{\mathcal{L}}(\ell_{123}, 0)$-module by Theorem 3.3 (or Proposition 4.5.17 in [24]). Since $M_{\mathcal{L}}(\ell_{123}, h_1, h_2)_{(h_1)} (= \mathbb{C}_{\ell_{123}, h_{12}})$ generates $M_{\mathcal{L}}(\ell_{123}, h_1, h_2)$, for any proper submodule U, U is graded by Theorem 3.3, and

$$U_{(h_1)} = U \cap M_{\mathcal{L}}(\ell_{123}, h_1, h_2)_{(h_1)} = 0.$$

Let $T_{\mathcal{L}}(\ell_{123}, h_1, h_2)$ be the sum of all the proper $\mathcal{L}$-submodules of the modul $M_{\mathcal{L}}(\ell_{123}, h_1, h_2)$, it is also graded. Then $T_{\mathcal{L}}(\ell_{123}, h_1, h_2)_{(h_1)} = 0$. So $T_{\mathcal{L}}(\ell_{123}, h_1, h_2)$ is also proper and is the largest proper submodule. Set

$$L_{\mathcal{L}}(\ell_{123}, h_1, h_2) = M_{\mathcal{L}}(\ell_{123}, h_1, h_2)/T_{\mathcal{L}}(\ell_{123}, h_1, h_2), \tag{3.4}$$

then $L_{\mathcal{L}}(\ell_{123}, h_1, h_2)$ is an irreducible $\mathcal{L}$-module.

By Theorem 3.3, $T_{\mathcal{L}}(\ell_{123}, h_1, h_2)$ is also the (unique) largest proper $V_{\mathcal{L}}(\ell_{123}, 0)$-submodule of $M_{\mathcal{L}}(\ell_{123}, h_1, h_2)$, so that $L_{\mathcal{L}}(\ell_{123}, h_1, h_2)$ is an irreducible $V_{\mathcal{L}}(\ell_{123}, 0)$-module.

THEOREM 3.6. *For any complex numbers $\ell_1, \ell_2, \ell_3, h_1, h_2$, $L_{\mathcal{L}}(\ell_{123}, h_1, h_2)$ is an irreducible module for the vertex operator algebra $V_{\mathcal{L}}(\ell_{123}, 0)$. Furthermore, the modules $L_{\mathcal{L}}(\ell_{123}, h_1, h_2)$ for $h_1, h_2 \in \mathbb{C}$ exhaust those irreducible (vertex operator algebra) $V_{\mathcal{L}}(\ell_{123}, 0)$-modules.*

PROOF. The first assertion has been showed above. For the second assertion, let $W = \coprod_{r \in \mathbb{C}} W_{(r)}$ be an irreducible $V_{\mathcal{L}}(\ell_{123}, 0)$-module, since I_0 is in the center, so it must acts on the irreducible module as a scalar, say I_0 acts on W as a scalar h_2. By Theorem 3.3, W must be of level ℓ_{123}, i.e. c_i acts on W as a scalar ℓ_i for $i = 1, 2, 3$. From (4.1.22) of [24], there exists $h_1 \in \mathbb{C}$ such that $W_{(h_1)} \neq 0$ and $W_{(h_1 - n)} = 0$ for all positive integers n. Let $0 \neq v \in W_{(h_1)}$. Then $L_0 v = h_1 v$, $L_n v = 0$ and $I_n v = 0$ for $n \geq 1$ since $L_n v, I_n v \in W_{(h_1 - n)}$. Hence by the universal property of $M_{\mathcal{L}}(\ell_{123}, h_1, h_2)$, there is a unique $\mathcal{L}$-module homomorphism

$$\psi : M_{\mathcal{L}}(\ell_{123}, h_1, h_2) \longrightarrow W;\ \mathbf{1}_{(\ell_{123}, h_{12})} \mapsto v.$$

By Proposition 4.5.1 of [24], ψ is a $V_{\mathcal{L}}(\ell_{123}, 0)$-module homomorphism (since $L_{-2}\mathbf{1}$ and $I_{-1}\mathbf{1}$ generates $V_{\mathcal{L}}(\ell_{123}, 0)$). Since W is irreducible and $T_{\mathcal{L}}(\ell_{123}, h_1, h_2)$ is the (unique) largest proper submodule of $M_{\mathcal{L}}(\ell_{123}, h_1, h_2)$, it follows that $\psi(M_{\mathcal{L}}(\ell_{123}, h_1, h_2)) = W$ and that $Ker\ \psi = T_{\mathcal{L}}(\ell_{123}, h_1, h_2)$. Thus ψ reduces to a $V_{\mathcal{L}}(\ell_{123}, 0)$-module isomorphism from $L_{\mathcal{L}}(\ell_{123}, h_1, h_2)$ onto W. ∎

REMARK 3.7. Similarly as in the case of $V_{\mathcal{L}}(\ell_{123}, 0)$, one can show that there is a vertex operator algebra structure on $V_{\mathfrak{L}}(\ell_{123}, 0)$, with $\overline{L}_{-1}\mathbf{1}$ a conformal vector,

$$V_{\mathfrak{L}}(\ell_{123}, 0) = \coprod_{n \geq 0} V_{\mathfrak{L}}(\ell_{123}, 0)_{(n)},$$

where $V_{\mathfrak{L}}(\ell_{123},0)_{(0)}=\mathbb{C}_{\ell_{123}}$ and $V_{\mathfrak{L}}(\ell_{123},0)_{(n)}$, $n\geq 1$, has a basis consisting of the vectors

$$\overline{I}_{-k_1}\cdots\overline{I}_{-k_s}\overline{L}_{-m_1}\cdots\overline{L}_{-m_r}\mathbf{1}$$

for $r,s\geq 0$, $m_1\geq\cdots\geq m_r\geq 1$, $k_1\geq\cdots\geq k_s\geq 1$ with

$$\sum_{i=1}^{r}(m_i+1)+\sum_{j=1}^{s}k_j=n.$$

Actually $V_{\mathfrak{L}}(\ell_{123},0)$ and $V_{\mathcal{L}}(\ell_{123},0)$ are isomorphic as vertex operator algebras.

In the following subsections, we will study the structure theory of the vertex operator algebra $V_{\mathcal{L}}(\ell_{123},0)$ and its simple descendant.

3.2. *Zhu's algebra, C_2-cofiniteness, rationality and regularity.* Recall the following notions, see for example [11, 12, 36] for detail.

Definition 3.8. *A vertex operator algebra V is called* C_2-cofinite *if $dim V/C_2(V)<\infty$, where $C_2(V)=span\{u_{-2}v\mid u,v\in V\}$.*

Definition 3.9. *A vertex operator algebra V is called* rational *if every admissible module is a direct sum of simple admissible modules.*

Definition 3.10. *A vertex operator algebra V is called* regular *if every weak module is a direct sum of simple ordinary modules.*

For any vertex operator algebra V, its *Zhu's algebra* is defined to be $A(V)=V/O(V)$, where $O(V)$ is the subspace of V spanned by elements

$$\{Res_z(Y(a,z)\frac{(1+z)^{wt\ a}}{z^2}b)\mid a,b\in V, a \text{ homogeneous}\},$$

note

$$Res_z(Y(a,z)\frac{(1+z)^{wt\ a}}{z^2}b)=\sum_{i\geq 0}\binom{wt\ a}{i}a_{i-2}b,$$

with the bilinear operation $*$ on V defined by

$$a*b=Res_z(Y(a,z)\frac{(1+z)^{wt\ a}}{z}b)=\sum_{i\geq 0}\binom{wt\ a}{i}a_{i-1}b\ \text{ for } a \text{ homogeneous}.$$

For $v\in V$, we denote the image of v in $A(V)$ by $[v]$, then

$$[a]*[b]=\sum_{i\geq 0}\binom{wt\ a}{i}[a_{i-1}b]\ \text{ for } a \text{ homogeneous}.$$

$(A(V),*)$ is an associative algebra with identity $[\mathbf{1}]$ ([36]).

For any $u\in V$, denote by $o(u)=u_{wt\ u-1}$. We do not recall the correspondence between V-module and $A(V)$-module here, the readers can check [36] for detail. As we need, we write down the following results.

LEMMA 3.11. *For homogeneous elements $a, b \in V$, and $m \geq n \geq 0$,*

$$Res_z(Y(a,z)\frac{(z+1)^{wt\ a+n}}{z^{2+m}}b) \in O(V).$$

LEMMA 3.12. *1. $o(u) = 0$ for any $u \in O(V)$;*
2. $o(u * v) = o(u)o(v)$;
3. $[u] \in A(V)$ acts on $A(V)$-module corresponds to the $o(u)$-action on the corresponding V-module.

Since I_0 commutes with L_m, I_n for any $m, n \in \mathbb{Z}$, we have

$$I_0 v = 0, \text{ hence } [I_0 v] = [0], \text{ for any } v \in V_{\mathcal{L}}(\ell_{123}, 0).$$

For any vertex operator algebra V with conformal vector (denoted by) ω, $[\omega]$ is in the center of $A(V)$. Hence for $V_{\mathcal{L}}(\ell_{123}, 0)$, $[\omega] * [I] = [I] * [\omega]$, i.e. $[\omega]$ commutes with $[I]$.

THEOREM 3.13. *There exists an isomorphism of associative algebras*

$$\varphi : \mathbb{C}[x,y] \longrightarrow A(V_{\mathcal{L}}(\ell_{123}, 0));\ 1 \mapsto [\mathbf{1}],\ x \mapsto [\omega],\ y \mapsto [I].$$

PROOF. Let A be the subalgebra of $A(V_{\mathcal{L}}(\ell_{123}, 0))$ generated by $[\omega]$, $[I]$ and $[\mathbf{1}]$. For the existence of the above surjective algebra homomorphism, it suffices to show that for every homogeneous $u \in V_{\mathcal{L}}(\ell_{123}, 0)$ with $wt\ u \geq 1$, we have $[u] \in A$. We show it by induction on $wt\ u$, note $wt\ u \geq 1$. For $u \in V_{\mathcal{L}}(\ell_{123}, 0)$ with $wt\ u = 1$, we only have one choice, i.e. $u = I$. Clearly $[u] = [I] \in A$. Suppose for all homogeneous u with $wt\ u \leq m - 1$ we have $[u] \in A$, then for u with $wt\ u = m$, we may assume that

$$u = I_{-k_1} \cdots I_{-k_s} L_{-m_1} \cdots L_{-m_r} \mathbf{1}$$

for $r, s \geq 0$, $m_1 \geq \cdots \geq m_r \geq 2$, $k_1 \geq \cdots \geq k_s \geq 1$ with $\sum_{i=1}^{r} m_i + \sum_{j=1}^{s} k_j = m$. If all k_j's are zero, i.e.

$$u = L_{-m_1} \cdots L_{-m_r} \mathbf{1},$$

then one get from Lemma 4.1 of [35] that $[u]$ can be generated by $[\omega]$. Otherwise, denote

$$u' = I_{-k_2} \cdots I_{-k_s} L_{-m_1} \cdots L_{-m_r} \mathbf{1},$$

then by induction $[u']$ is in $A(V_{\mathcal{L}}(\ell_{123}, 0))$. And $u = I_{-k_1} u'$, $k_1 \geq 1$. By Lemma 3.11, take $a = I$, $n = 0$, note $wt\ a = 1$, then we have

$$[(I_{-n} + I_{-n-1})b] = [0] \text{ for } b \text{ homogeneous},\ n \geq 1.$$

So without loss of generality, we may assume $k_1 = 1$, i.e. $u = I_{-1} u'$. Then

$$[I] * [u'] = [(I_{-1} + I_0)u'] = [I_{-1} u'] + [I_0 u'] = [u].$$

With $[u'], [I] \in A$ we get $[u] \in A$.

We now show that φ is injective. Assume $0 \neq f(x,y) \in Ker(\varphi)$, we can write it as $f(x,y) = \sum_{f.s.} a_{mn} x^m y^n$ (f.s. means finite sum), then

$\sum_{f.s.} a_{mn}[\omega]^m[I]^n = [0]$ in $A(V_{\mathcal{L}}(\ell_{123}, 0))$, and so $\sum_{f.s.} a_{mn}[\omega]^m[I]^n$ acts trivially on any $A(V_{\mathcal{L}}(\ell_{123}, 0))$-module which corresponds to that $\sum_{f.s.} a_{mn}o(\omega)^m o(I)^n$ acts as zero on the bottom level of any $V_{\mathcal{L}}(\ell_{123}, 0)$-module. Note $o(\omega) = \omega_1 = L_0$ and $o(I) = I_0$. Recall for any $h_1, h_2 \in \mathbb{C}$, $M_{\mathcal{L}}(\ell_{123}, h_1, h_2)$ is a module for $V_{\mathcal{L}}(\ell_{123}, 0)$, with the bottom level $\mathbb{C}_{\ell_{123}, h_{12}}$, and L_0 acts as h_1, I_0 acts as h_2, hence on $\mathbb{C}_{\ell_{123}, h_{12}}$ we have

$$0 = \sum_{f.s.} a_{mn}o(\omega)^m o(I)^n = \sum_{f.s.} a_{mn}L_0^m I_0^n = \sum_{f.s.} a_{mn}h_1^m h_2^n.$$

Clearly, there exist elements h_1, h_2 such that $\sum_{f.s.} a_{mn}h_1^m h_2^n \neq 0$, contradiction. Hence φ is injective. $\square$

Remark 3.14. Later on, we will characterize $V_{\mathcal{L}}(\ell_{123}, 0)$ as a tensor product of two vertex operator algebras, which immediately gives that $A(V_{\mathcal{L}}(\ell_{123}, 0))$ is isomorphic to a polynomial algebra over $\mathbb{C}$ with two variables. But our proof above is more intrinsic and gives expectation (or some sense) that $V_{\mathcal{L}}(\ell_{123}, 0)$ may be isomorphic to a tensor product of two vertex operator algebras.

For a vertex operator algebra to be regular, rational and C_2-cofinite, its Zhu's algebra must be of finite dimensional (c.f. [36, 12]). Now $A(V_{\mathcal{L}}(\ell_{123}, 0)) \cong \mathbb{C}[x, y]$ is an infinite dimensional $\mathbb{C}$-algebra, hence we have:

Proposition 3.15. *$V_{\mathcal{L}}(\ell_{123}, 0)$ is not regular, not rational and not C_2-cofinite.*

3.3. *Commutant.* We now look at the commutant of Heisenberg vertex operator algebra in $V_{\mathcal{L}}(\ell_{123}, 0)$. Recall that if $(V, Y, \mathbf{1}, \omega)$ is a vertex operator algebra and $(U, Y, \mathbf{1}, \omega')$ is a vertex operator subalgebra of V, then the *commutant* is defined to be

$$U^c = \{v \in V \mid L'(-1)v = 0\},$$

where $L'(-1)$ is determined by ω'.

For our $V_{\mathcal{L}}(\ell_{123}, 0)$, we know that when $\ell_3 \neq 0$, $U = V_{\mathcal{H}}(\ell_3, 0)$ is a vertex operator algebra with standard conformal vector $\omega' = \frac{1}{2\ell_3} I_{-1}I_{-1}\mathbf{1}$ (of central charge 1).

$$L_1\omega' = \frac{1}{2\ell_3} L_1 I_{-1} I_{-1}\mathbf{1} = \frac{-2\ell_2}{\ell_3} I_{-1}\mathbf{1},$$

so $L_1\omega' = 0$ if and only if $\ell_2 = 0$. Hence when $\ell_2 = 0$, by Theorem 5.1 of [15], the commutant U^c of the Heisenberg vertex operator algebra $U = V_{\mathcal{H}}(\ell_3, 0)$ (equipped with the standard conformal vector ω') is a vertex operator algebra with conformal element $\omega'' = \omega - \omega'$.

The next thing we want to do is to characterize the generators (or even basis) of the commutant (actually, we get a more powerful result, see below). We consider $\ell_3 \neq 0$ in the following for necessity.

For the basic notions and results of tensor product vertex operator algebra, see for example [24]. The idea of the following theorem comes from the paper [6].

THEOREM 3.16. *When* $\ell_3 \neq 0$, $V_{\mathcal{L}}(\ell_{123}, 0)$ *is isomorphic to the tensor product* $V_{\mathcal{H}}(\ell_3, 0) \otimes V_{\tilde{Vir}}(c_{\tilde{Vir}}, 0)$ *as vertex operator algebras, where* $V_{\mathcal{H}}(\ell_3, 0)$ *is equipped with a nonstandard conformal vector* $\omega_H = \omega' + \frac{\ell_2}{\ell_3} I_{-2}\mathbf{1}$ *(of central charge* $1 - 12\frac{\ell_2^2}{\ell_3}$*), and* $\tilde{Vir}$ *is a new Virasoro algebra constructed in the proof with central charge* $c_{\tilde{Vir}} = \ell_1 - 1 + 12\frac{\ell_2^2}{\ell_3}$.

PROOF. Firstly, denote $\tilde{\omega} = \omega - \omega_H$, it is straightforward to show that ω_H and $\tilde{\omega}$ satisfy Virasoro algebra relations. So $\tilde{\omega}$ gives a Virasoro algebra which we denoted by $\tilde{Vir}$ and from $\tilde{Vir}$ we get a vertex operator algebra which we denoted by $V_{\tilde{Vir}}(c_{\tilde{Vir}}, 0)$, it is with conformal vector $\tilde{\omega}$ and is of central charge $c_{\tilde{Vir}} = \ell_1 - 1 + 12\frac{\ell_2^2}{\ell_3}$. Since $(I_{-2}\mathbf{1})_0 = 0$ and $(I_{-2}\mathbf{1})_1 = \omega_1' - I_0$, I_0 acts on $V_{\mathcal{H}}(\ell_3, 0)$ as zero and ω' is a conformal vector, we see that ω_H is also a conformal vector of $V_{\mathcal{H}}(\ell_3, 0)$ and is of central charge $1 - 12\frac{\ell_2^2}{\ell_3}$. Next, it is straightforward to show that

$$\tilde{\omega}_n I_m \mathbf{1} = I_m \tilde{\omega}_n \mathbf{1}$$

for any $m, n \in \mathbb{Z}$. At last, we define a map

$$\varphi : V_{\mathcal{H}}(\ell_3, 0) \otimes V_{\tilde{Vir}}(c_{\tilde{Vir}}, 0) \longrightarrow V_{\mathcal{L}}(\ell_{123}, 0)$$

on the basis by

$$I_{-k_1} \cdots I_{-k_s}\mathbf{1} \otimes \tilde{\omega}_{-m_1} \cdots \tilde{\omega}_{-m_r}\mathbf{1} \mapsto I_{-k_1} \cdots I_{-k_s}\tilde{\omega}_{-m_1} \cdots \tilde{\omega}_{-m_r}\mathbf{1}$$

and extend $\mathbb{C}$-linearly, it is easy to check that φ is a linear isomorphism, $\varphi(\mathbf{1} \otimes \mathbf{1}) = \mathbf{1}$ and $\varphi(\omega_H \otimes \mathbf{1} + \mathbf{1} \otimes \tilde{\omega}) = \tilde{\omega} + \omega_H = \omega$. It remains to show that it is a vertex algebra homomorphism. For this, note that

$$(I_{-k_1} \cdots I_{-k_s}\tilde{\omega}_{-m_1} \cdots \tilde{\omega}_{-m_r}\mathbf{1})_n = \sum_{i \in \mathbb{Z}} (I_{-k_1} \cdots I_{-k_s}\mathbf{1})_i (\tilde{\omega}_{-m_1} \cdots \tilde{\omega}_{-m_r}\mathbf{1})_{n-i-1}$$

on $V_{\mathcal{L}}(\ell_{123}, 0)$ for all $n \in \mathbb{Z}$. □

REMARK 3.17. The equation $\tilde{\omega}_n I_m \mathbf{1} = I_m \tilde{\omega}_n \mathbf{1}$ is important in proving that φ is a vertex algebra homomorphism. Note when $\ell_3 = 0$, $V_{\mathcal{H}}(\ell_3, 0)$ still has a vertex algebra structure, take $\tilde{\omega} = \omega$, so one may think about that just considering them as vertex algebras, can we characterize $V_{\mathcal{L}}(\ell_{123}, 0) \cong V_{\mathcal{H}}(0, 0) \otimes V_{Vir}(\ell_1, 0)$ as vertex algebras? The answer is not positive since now we do not have $\omega_n I_m \mathbf{1} = I_m \omega_n \mathbf{1}$ in general (even if you require $\ell_2 = 0$). The defined φ is a linear isomorphism for sure, but it may not be a vertex algebra homomorphism anymore.

By the above theorem, we get the following statement.

PROPOSITION 3.18. *For any $\ell_2 \in \mathbb{C}$, $\ell_1 \in \mathbb{C}$, $0 \neq \ell_3 \in \mathbb{C}$, the commutant of $V_{\mathcal{H}}(\ell_3, 0)$ (with conformal vector ω_H) in $V_{\mathcal{L}}(\ell_{123}, 0)$ is a vertex operator algebra and is isomorphic to the Virasoro vertex operator algebra $V_{\tilde{Vir}}(c_{\tilde{Vir}}, 0)$ (whose structure is clear).*

REMARK 3.19. Note in particular when $\ell_2 = 0$, $V_{\mathcal{H}}(\ell_3, 0)$ is with usual conformal vector $\omega_H = \omega^{'}$, so we've answered the commutant question that we originally considered.

REMARK 3.20. As explained before in Remark 3.14, with Theorem 3.16, by the fact that Zhu's algebra $A(V \otimes W) = A(V) \otimes A(W)$, and the result of Heisenberg and Virasoro vertex operator algebras, we can immediately have that $A(V_{\mathcal{L}}(\ell_{123}, 0))$ is isomorphic to a polynomial algebra in two variables.

3.4. *Simple vertex operator algebra and its structure.* We now look at the structure of the simple descendant of $V_{\mathcal{L}}(\ell_{123}, 0)$. The simple vertex operator algebra comes from $V_{\mathcal{L}}(\ell_{123}, 0)$ is of the form $L_{\mathcal{L}}(\ell_{123}, 0) = V_{\mathcal{L}}(\ell_{123}, 0)/T$, where T is the maximal ideal of $V_{\mathcal{L}}(\ell_{123}, 0)$ (which is equivalent to say that T is the maximal $\mathcal{L}$-submodule of the $\mathcal{L}$-module $V_{\mathcal{L}}(\ell_{123}, 0)$). We do not study the structure of the maximal submodule T directly, instead we use Theorem 3.16 to get a characterization of it and hence of $V_{\mathcal{L}}(\ell_{123}, 0)/T$.

A vertex operator algebra is simple if it is simple as a vertex algebra, so simplicity of a vertex operator algebra does not depend on the conformal vector. We know that $V_{\mathcal{H}}(\ell_3, 0)$ ($\ell_3 \neq 0$) with the usual conformal vector $\omega^{'}$ is simple (of central charge 1), so $V_{\mathcal{H}}(\ell_3, 0)$ with a nonstand conformal vector ω_H is also a simple vertex operator algebra (of central charge $1 - 12\frac{\ell_2^2}{\ell_3}$).

Denote by $c_{p,q} = 1 - 6\frac{(p-q)^2}{pq}$, where $p, q \in \{2, 3, 4, \ldots\}$ and are relatively prime, recall (c.f. [35, 24]) the following property.

PROPOSITION 3.21. *1. If $c_{\tilde{Vir}} \neq c_{p,q}$, then $V_{\tilde{Vir}}(c_{\tilde{Vir}}, 0)$ is a simple vertex operator algebra.*
2. If $c_{\tilde{Vir}} = c_{p,q}$, then $V_{\tilde{Vir}}(c_{\tilde{Vir}}, 0)$ has the maximal ideal $\langle v_{p,q} \rangle$ generated by a singular vector $v_{p,q}$ of degree $(p-1)(q-1)$.

By Corollary 4.7.3 of [13], tensor product of vertex operator algebras is simple if and only if each term is simple. So we have the statement.

PROPOSITION 3.22. *1. When $\ell_3 \neq 0$ and $c_{\tilde{Vir}} \neq c_{p,q}$, $V_{\mathcal{H}}(\ell_3, 0) \otimes V_{\tilde{Vir}}(c_{\tilde{Vir}}, 0)$ is a simple vertex operator algebra, and hence $V_{\mathcal{L}}(\ell_{123}, 0)$ is also a simple vertex operator algebra.*

2. *When $\ell_3 \neq 0$ and $c_{\tilde{Vir}} = c_{p,q}$, $V_{\tilde{Vir}}(c_{\tilde{Vir}}, 0)$ has the maximal ideal generated by the singular vector $v_{p,q}$, and the quotient $V_{\tilde{Vir}}(c_{\tilde{Vir}}, 0)/\langle v_{p,q}\rangle$ is a simple vertex operator algebra with conformal vector $\overline{\tilde{\omega}} = \tilde{\omega} + \langle v_{p,q}\rangle$, hence then $V_{\mathcal{H}}(\ell_3, 0) \otimes (V_{\tilde{Vir}}(c_{\tilde{Vir}}, 0)/\langle v_{p,q}\rangle)$ is a simple vertex operator algebra.*

And we have the following result.

Proposition 3.23. *For $c_{\tilde{Vir}} = c_{p,q}$ and $\ell_3 \neq 0$,*

1. *$V_{\mathcal{H}}(\ell_3, 0) \otimes \langle v_{p,q}\rangle$ is a proper ideal of the vertex operator algebra $V_{\mathcal{H}}(\ell_3, 0) \otimes V_{\tilde{Vir}}(c_{\tilde{Vir}}, 0)$.*
2. *We have*
$$\begin{aligned}(V_{\mathcal{H}}(\ell_3, 0) \otimes V_{\tilde{Vir}}(c_{\tilde{Vir}}, 0))/(V_{\mathcal{H}}(\ell_3, 0) \otimes \langle v_{p,q}\rangle)\\ \cong V_{\mathcal{H}}(\ell_3, 0) \otimes (V_{\tilde{Vir}}(c_{\tilde{Vir}}, 0)/\langle v_{p,q}\rangle)\end{aligned}$$
as vertex operator algebras. Hence $V_{\mathcal{H}}(\ell_3, 0) \otimes \langle v_{p,q}\rangle$ is the maximal ideal of $V_{\mathcal{H}}(\ell_3, 0) \otimes V_{\tilde{Vir}}(c_{\tilde{Vir}}, 0)$. So the simple vertex operator algebra $L_{\mathcal{L}}(\ell_{123}, 0)$ is isomorphic to the tensor product $V_{\mathcal{H}}(\ell_3, 0) \otimes (V_{\tilde{Vir}}(c_{\tilde{Vir}}, 0)/\langle v_{p,q}\rangle)$.

Proof. The first one can be proved by definition. For the second one, construct a map
$$\pi : V_{\mathcal{H}}(\ell_3, 0) \otimes V_{\tilde{Vir}}(c_{\tilde{Vir}}, 0) \longrightarrow V_{\mathcal{H}}(\ell_3, 0) \otimes (V_{\tilde{Vir}}(c_{\tilde{Vir}}, 0)/\langle v_{p,q}\rangle)$$
by $v \otimes w \mapsto v \otimes \overline{w}$ and extend linearly. Then clearly π is surjective with kernel $V_{\mathcal{H}}(\ell_3, 0) \otimes \langle v_{p,q}\rangle$, and so induces a desired linear isomorphism. It is also clear that the induced map takes the conformal vector to the conformal vector and is a vertex algebra homomorphism, hence it is a vertex operator algebra isomorphism. ∎

Therefore the structure of the simple vertex operator algebra is clear.

Theorem 3.24. 1. *When $\ell_3 \neq 0$ and $c_{\tilde{Vir}} \neq c_{p,q}$, $L_{\mathcal{L}}(\ell_{123}, 0) = V_{\mathcal{L}}(\ell_{123}, 0)$, so $T = 0$.*
2. *When $\ell_3 \neq 0$ and $c_{\tilde{Vir}} = c_{p,q}$,*
$$L_{\mathcal{L}}(\ell_{123}, 0) = V_{\mathcal{L}}(\ell_{123}, 0)/T \cong V_{\mathcal{H}}(\ell_3, 0) \otimes (V_{\tilde{Vir}}(c_{\tilde{Vir}}, 0)/\langle v_{p,q}\rangle),$$
where $T = \varphi(V_{\mathcal{H}}(\ell_3, 0) \otimes \langle v_{p,q}\rangle)$ for φ defined in Theorem 3.16.

Now for $\ell_3 \neq 0$ and $c_{\tilde{Vir}} = c_{p,q}$, Zhu's algebra of the simple vertex operator algebra $L_{\mathcal{L}}(\ell_{123}, 0)$ is isomorphic to $A\left(V_{\mathcal{H}}(\ell_3, 0) \otimes (V_{\tilde{Vir}}(c_{\tilde{Vir}}, 0)/\langle v_{p,q}\rangle)\right)$ which is equal to $A(V_{\mathcal{H}}(\ell_3, 0)) \otimes A(V_{\tilde{Vir}}(c_{\tilde{Vir}}, 0)/\langle v_{p,q}\rangle)$. So $A(L_{\mathcal{L}}(\ell_{123}, 0))$ is isomorphic to $\mathbb{C}[x] \otimes (\mathbb{C}[y]/(G_{p,q}(y)))$ (where $(G_{p,q}(y))$ is a polynomial of degree $\frac{1}{2}(p-1)(q-1)$) (c.f. [35, 15]), which is infinite dimensional. Therefore, the simple vertex operator algebra $L_{\mathcal{L}}(\ell_{123}, 0)$ is also not regular, not C_2-cofinite and not rational.

3.5. *Unitary vertex operator algebra and its unitary module.* At the end of this section, we look at the unitary structure of the simple vertex operator algebra $L_{\mathcal{L}}(\ell_{123}, 0)$, we follow some related notions by [9].

DEFINITION 3.25. *A vertex operator algebra $(V, Y, \mathbf{1}, \omega)$ is called* CFT-type *if $V_n = 0$ for $n < 0$ and $V_0 = \mathbb{C}\mathbf{1}$.*

Clearly, our $L_{\mathcal{L}}(\ell_{123}, 0)$ is of CFT-type.

DEFINITION 3.26. *Let $(V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra of CFT-type. An* anti-linear automorphism *ϕ of V is an anti-linear map $\phi : V \longrightarrow V$ such that $\phi(\mathbf{1}) = \mathbf{1}$, $\phi(\omega) = \omega$ and $\phi(u_n v) = \phi(u)_n \phi(v)$ for any $u, v \in V$ and $n \in \mathbb{Z}$.*

DEFINITION 3.27. *Let $(V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra of CFT-type and $\phi : V \longrightarrow V$ be an anti-linear involution, i.e. an anti-linear automorphism of order 2. Then (V, ϕ) is called* unitary *if there exists a positive definite Hermitian form $(,) : V \times V \longrightarrow \mathbb{C}$ which is $\mathbb{C}$-linear on the first vector and anti-$\mathbb{C}$-linear on the second vector such that the following invariant property holds: for any $a, u, v \in V$*

$$(Y(e^{zL(1)}(-z^{-2})^{L(0)}a, z^{-1})u, v) = (u, Y(\phi(a), z)v) \tag{3.5}$$

where $L(n)$ is defined by $Y(\omega, z) = \sum_{n\in\mathbb{Z}} L(n) z^{-n-2}$.

DEFINITION 3.28. *Let $(V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra of CFT-type and $\phi : V \longrightarrow V$ be an anti-linear involution, A V-module (M, Y_M) is called a* unitary V-module *if there exists a positive definite Hermitian form $(,)_M : M \times M \longrightarrow \mathbb{C}$ which is $\mathbb{C}$-linear on the first vector and anti-$\mathbb{C}$-linear on the second vector such that the following invariant property holds:*

$$(Y_M(e^{zL(1)}(-z^{-2})^{L(0)}a, z^{-1})w_1, w_2)_M = (w_1, Y_M(\phi(a), z)w_2)_M \tag{3.6}$$

for any $a \in V$, $w_1, w_2 \in M$.

REMARK 3.29. Unitarity is not an isomorphic invariant property of vertex operator algebras. For example, $V_{\mathcal{H}}(\ell_3, 0)$ is isomorphic to $V_{\mathcal{H}}(1, 0)$ as vertex operator algebras for any $0 \neq \ell_3 \in \mathbb{C}$, $V_{\mathcal{H}}(1, 0)$ is a unitary vertex operator algebra ([9]), but of course it is not that for any $0 \neq \ell_3 \in \mathbb{C}$, $V_{\mathcal{H}}(\ell_3, 0)$ is a unitary vertex operator algebra (for example, the positive definiteness of the Hermitian form requires $\ell_3 \in \mathbb{R}_{>0}$). So even though tensor product of unitary vertex operator algebras is unitary (Proposition 2.9 of [9]), and we know (Theorem 3.16, Theorem 3.24) that our $L_{\mathcal{L}}(\ell_{123}, 0)$ is isomorphic to a tensor product of two unitary vertex operator algebras under certain conditions (c.f. [9]), we can't say that $V_{\mathcal{L}}(\ell_{123}, 0)$ is a unitary vertex operator algebra under those conditions.

Define an anti-linear map ϕ of $V_{\mathcal{L}}(\ell_{123}, 0)$ as follows:

$$\phi : V_{\mathcal{L}}(\ell_{123}, 0) \longrightarrow V_{\mathcal{L}}(\ell_{123}, 0),$$

$$I_{-k_1}\cdots I_{-k_s}L_{-m_1}\cdots L_{-m_r}\mathbf{1}\mapsto(-1)^sI_{-k_1}\cdots I_{-k_s}L_{-m_1}\cdots L_{-m_r}\mathbf{1},$$

where $r,s\geq 0$, $m_1\geq\cdots\geq m_r\geq 2$, $k_1\geq\cdots\geq k_s\geq 1$.

LEMMA 3.30. *ϕ is an anti-linear involution of vertex operator algebra $V_{\mathcal{L}}(\ell_{123},0)$. Furthermore, ϕ induces an anti-linear involution (also denoted by ϕ) of $L_{\mathcal{L}}(\ell_{123},0)$.*

PROOF. $\phi^2=id$, so it is enough to show that ϕ is an anti-linear automorphism. By definition of our ϕ above, we have $\phi(\mathbf{1})=\mathbf{1}$, $\phi(\omega)=\omega$ and $\phi(I)=-I$. Let

$$U=\{u\in V_{\mathcal{L}}(\ell_{123},0)\mid\phi(u_nv)=\phi(u)_n\phi(v),\forall\, v\in V_{\mathcal{L}}(\ell_{123},0),n\in\mathbb{Z}\},$$

then U is a subspace of $V_{\mathcal{L}}(\ell_{123},0)$. It is straightforward to show that if $a,b\in U$, then $a_mb\in U$ for any $m\in\mathbb{Z}$. Now we show that the generators ω and I of $V_{\mathcal{L}}(\ell_{123},0)$ are in U. For any $v\in V_{\mathcal{L}}(\ell_{123},0),n\in\mathbb{Z}$,

$$\phi(\omega_nv)=\phi(L_{n-1}v)=L_{n-1}\phi(v)=\omega_n\phi(v)=(\phi(\omega))_n\phi(v),$$

$$\phi((I)_nv)=\phi(I_nv)=-I_n\phi(v)=\phi(I)_n\phi(v),$$

so $\omega,I\in U$, and then we have $V_{\mathcal{L}}(\ell_{123},0)=U$. Thus ϕ is an anti-linear involution of $V_{\mathcal{L}}(\ell_{123},0)$.

Let T be the maximal proper $\mathcal{L}$-submodule of $V_{\mathcal{L}}(\ell_{123},0)$, we have $\phi(T)$ is a proper $\mathcal{L}$-submodule of $V_{\mathcal{L}}(\ell_{123},0)$, so $\phi(T)\subseteq T$. Hence ϕ induces an anti-linear involution of $L_{\mathcal{L}}(\ell_{123},0)$. ∎

Note that for the two Lie algebras $\widehat{\mathfrak{D}}$ and $\mathcal{L}$ defined in [2] and [5] respectively, there exists a Lie algebra isomorphism between them (for notations related to $\widehat{\mathfrak{D}}$ we follow [2])

$$\rho:\mathcal{L}\longrightarrow\widehat{\mathfrak{D}};$$
$$L_m\mapsto d_m,\ c_1\mapsto c,\ c_3\mapsto c_a,\ I_n\mapsto z^n-\sqrt{-1}\delta_{n,0}c_3,\ c_2\mapsto-\sqrt{-1}c_3,$$

which then induces an isomorphism between the Verma modules (as $\mathcal{L}$-modules and as $\widehat{\mathfrak{D}}$-modules via ρ)

$$M_{\mathcal{L}}(\ell_{123},h_1,h_2)\cong\widetilde{R}(\ell_1,h_1,\ell_3,h_2-\ell_2,\sqrt{-1}\ell_2),$$

and hence an isomorphism of irreducible $\mathcal{L}$-modules

$$L_{\mathcal{L}}(\ell_{123},h_1,h_2)\cong R(\ell_1,h_1,\ell_3,h_2-\ell_2,\sqrt{-1}\ell_2).$$

In particular,

$$L_{\mathcal{L}}(\ell_{123},0)\cong R(\ell_1,0,\ell_3,-\ell_2,\sqrt{-1}\ell_2)$$

as irreducible $\mathcal{L}$-modules, where as irreducible $\mathcal{L}$-modules $L_{\mathcal{L}}(\ell_{123},0)=L_{\mathcal{L}}(\ell_{123},0,0)$.

And via the isomorphism ρ, the anti-linear anti-involution $*$ defined in [2] induces an anti-linear anti-involution σ (see also [5]) on $\mathcal{L}$ defined as

$$\sigma(L_n) = L_{-n}, \ \ \sigma(I_n) = I_{-n} - 2\delta_{n,0}c_2,$$
$$\sigma(c_1) = c_1, \ \ \sigma(c_3) = c_3, \ \ \sigma(c_2) = -c_2.$$

Then the unique contravariant (under $*$) Hermitian form in Section 6 of [2] induces a unique Hermitian form $(\ , \ )$ on $L_{\mathcal{L}}(\ell_{123}, h_1, h_2)$ such that $(w, w) = 1$ for a highest weight vector w and is contravariant with respect to the anti-involution σ, i.e.

$$(xu, v) = (u, \sigma(x)v), \ \ x \in \mathcal{L}, \ \ u, v \in L_{\mathcal{L}}(\ell_{123}, h_1, h_2).$$

In particular,

$$(I_n u, v) = (u, \sigma(I_n)v), \ \ (L_n u, v) = (u, \sigma(L_n)v), \ \ u, v \in L_{\mathcal{L}}(\ell_{123}, h_1, h_2).$$

Denote by $c_m = 1 - \frac{6}{m(m+1)},$, $h^m_{r,s} = \frac{(r(m+1)-sm)^2-1}{4m(m+1)}$, $1 \leq s \leq r \leq m-1$, $m \in \mathbb{Z}_{\geq 2}$. Then for the positive definiteness of the Hermitian form, by Theorem 6.6 of [2] we get the following statement.

Proposition 3.31. *The contravariant Hermitian form $(\ , \ )$ on the module $L_{\mathcal{L}}(\ell_{123}, h_1, h_2)$ is positive definite in precisely the following cases:*

1. *$\ell_3 = 0$, then $\ell_2 = 0, h_2 = 0$, and $\ell_1 \in \mathbb{R}_{\geq 1}, h_1 \in \mathbb{R}_{\geq 0}$ or $\ell_1 = c_m, h_1 = h^m_{r,s}$, $1 \leq s \leq r \leq m-1$, $m \in \mathbb{Z}_{\geq 2}$;*
2. *$\ell_3 \in \mathbb{R}_{>0}$, then either*

$$\ell_1 - 12\frac{\ell_2^2}{\ell_3} \in \mathbb{R}_{\geq 2}, \ h_1 - \frac{(h_2 - \ell_2)^2 + (\sqrt{-1}\ell_2)^2}{2\ell_3} \in \mathbb{R}_{\geq 0},$$

or

$$\ell_1 - 12\frac{\ell_2^2}{\ell_3} = 1 + c_m, \ h_1 - \frac{(h_2 - \ell_2)^2 + (\sqrt{-1}\ell_2)^2}{2\ell_3} = h^m_{r,s},$$

where $1 \leq s \leq r \leq m-1$, $m \in \mathbb{Z}_{\geq 2}$.

For the unitary property of $L_{\mathcal{L}}(\ell_{123}, 0)$ we need to show (3.5) holds, and we only need to show this for the generators I, ω of $L_{\mathcal{L}}(\ell_{123}, 0)$ by Proposition 2.11 of [9]. For ω we can show in the same way as Theorem 4.2 of [9]. For I, we need to do some more work since the anti-involution now is defined differently. For any $u, v \in L_{\mathcal{L}}(\ell_{123}, 0)$,

$$\begin{aligned}
&\big(Y(e^{zL_1}(-z^{-2})^{L_0}I, z^{-1})u, v\big) \\
&\quad = \big(Y(-z^{-2}I, z^{-1})u, v\big) = \sum_{n\in\mathbb{Z}} -(I_n u, v)z^{n-1} \\
&\quad = \sum_{n\neq 0} -(I_n u, v)z^{n-1} + (-(I_0 u, v)z^{-1})
\end{aligned}$$

$$= \sum_{n\neq 0} -(u, I_{-n}v)z^{n-1} - (u, I_0 v)z^{-1} + 2\ell_2(u,v)z^{-1}$$
$$= -\sum_{n\in\mathbb{Z}} (u, I_{-n}v)z^{n-1} + 2\ell_2(u,v)z^{-1}$$
$$= (u, Y(\phi(I), z)v) + 2\ell_2(u,v)z^{-1}.$$

In order that (3.5) holds for I, we need to require that $\ell_2 = 0$.

The above shows that the following theorem holds.

THEOREM 3.32. *$(L_{\mathcal{L}}(\ell_{123}, 0), \phi)$ is a unitary vertex operator algebra if and only if one of the following holds:*

1. *$\ell_2 = 0, \ell_3 = 0$, $\ell_1 \in \mathbb{R}_{\geq 1}$ or $\ell_1 = c_m, m \in \mathbb{Z}_{\geq 2}$;*
2. *$\ell_2 = 0, \ell_3 \in \mathbb{R}_{>0}$, $\ell_1 \in \mathbb{R}_{\geq 2}$ or $\ell_1 = 1 + c_m, m \in \mathbb{Z}_{\geq 2}$.*

Similar to the proof of Theorem 3.32, we can get the following statement.

THEOREM 3.33. *Let ϕ be defined as above. Then $L_{\mathcal{L}}(\ell_{123}, h_1, h_2)$ is a unitary module of $(L_{\mathcal{L}}(\ell_{123}, 0), \phi)$ if and only if one of the following holds:*

1. *$\ell_2 = 0$, $\ell_3 = 0$, $h_2 = 0$, $\ell_1 \in \mathbb{R}_{\geq 1}, h_1 \in \mathbb{R}_{\geq 0}$ or $\ell_1 = c_m, h_1 = h^m_{r,s}$, $1 \leq s \leq r \leq m-1$, $m \in \mathbb{Z}_{\geq 2}$;*
2. *$\ell_2 = 0$, $\ell_3 \in \mathbb{R}_{>0}$, $\ell_1 \in \mathbb{R}_{\geq 2}$, $h_1 - \frac{(h_2)^2}{2\ell_3} \in \mathbb{R}_{\geq 0}$, or $\ell_1 = 1 + c_m$, $h_1 - \frac{(h_2)^2}{2\ell_3} = h^m_{r,s}$, $1 \leq s \leq r \leq m-1$, $m \in \mathbb{Z}_{\geq 2}$.*

4. RANK TWO CASE WITH ϕ-COORDINATED MODULES

In this section, we associate the rank two twisted Heisenberg-Virasoro algebra $\mathcal{L}^*$ with the vertex algebra $V_{\widehat{\mathfrak{L}^*}}(\ell_{1234}, 0)$ in terms of its ϕ-coordinated module. More specifically, we show that there is a one-to-one correspondence between restricted $\mathcal{L}^*$-modules of *level* ℓ_{1234} and ϕ-coordinated modules for the vertex algebra $V_{\widehat{\mathfrak{L}^*}}(\ell_{1234}, 0)$, where $\widehat{\mathfrak{L}^*}$ is a newly defined Lie algebra.

We first recall from [33] the definition of rank two twisted Heisenberg-Virasoro algebra.

DEFINITION 4.1. *The rank two twisted Heisenberg-Virasoro algebra $\mathcal{L}^*$ is a vector space spanned by $t_1^m t_2^n, E_{m,n}, K_i, i = 1, 2, 3, 4$, for $m, n \in \mathbb{Z}^2\backslash\{(0,0)\}$, with the following Lie brackets:*

$$[t_1^m t_2^n, t_1^r t_2^s] = 0; \quad [K_i, \mathcal{L}^*] = 0,\ i = 1, 2, 3, 4;$$
$$[t_1^m t_2^n, E_{r,s}] = (nr - ms)t_1^{m+r}t_2^{n+s} + \delta_{m+r,0}\delta_{n+s,0}(mK_1 + nK_2);$$
$$[E_{m,n}, E_{r,s}] = (nr - ms)E_{m+r,n+s} + \delta_{m+r,0}\delta_{n+s,0}(mK_3 + nK_4),$$

for $(m, n), (r, s) \in \mathbb{Z}^2\backslash\{(0,0)\}$.

We form the generating functions as

$$T_m(x) = \sum_{n\in\mathbb{Z}} t_1^m t_2^n x^{-n}, \quad E_m(x) = \sum_{n\in\mathbb{Z}} E_{m,n} x^{-n}. \tag{4.1}$$

Rewriting the Lie brackets as

$$[t_1^m t_2^n, E_{r,s}] = (n(m+r) - m(n+s))\, t_1^{m+r} t_2^{n+s} + \delta_{m+r,0}\delta_{n+s,0}(mK_1 + nK_2),$$
$$[E_{m,n}, E_{r,s}] = (n(m+r) - m(n+s))\, E_{m+r,n+s} + \delta_{m+r,0}\delta_{n+s,0}(mK_3 + nK_4),$$

we can get the following generating function brackets:

$$\begin{aligned}
&[T_m(x_1), E_r(x_2)]\\
&\quad = (m+r)T_{m+r}(x_2)\left(x_2\frac{\partial}{\partial x_2}\right)\delta\left(\frac{x_2}{x_1}\right)\\
&\qquad + m\left(x_2\frac{\partial}{\partial x_2}T_{m+r}(x_2)\right)\delta\left(\frac{x_2}{x_1}\right)\\
&\qquad + m\delta_{m+r,0}\delta\left(\frac{x_2}{x_1}\right)K_1 + \delta_{m+r,0}\left(x_2\frac{\partial}{\partial x_2}\right)\delta\left(\frac{x_2}{x_1}\right)K_2,
\end{aligned}\tag{4.2}$$

and

$$\begin{aligned}
&[E_m(x_1), E_r(x_2)]\\
&\quad = (m+r)E_{m+r}(x_2)\left(x_2\frac{\partial}{\partial x_2}\right)\delta\left(\frac{x_2}{x_1}\right)\\
&\qquad + m\left(x_2\frac{\partial}{\partial x_2}E_{m+r}(x_2)\right)\delta\left(\frac{x_2}{x_1}\right)\\
&\qquad + m\delta_{m+r,0}\delta\left(\frac{x_2}{x_1}\right)K_3 + \delta_{m+r,0}\left(x_2\frac{\partial}{\partial x_2}\right)\delta\left(\frac{x_2}{x_1}\right)K_4.
\end{aligned}\tag{4.3}$$

Definition 4.2. *An $\mathcal{L}^*$-module W is said to be* restricted *if for any $w \in W, (m,n),(r,s) \in \mathbb{Z}^2\backslash\{(0,0)\}$, $t_1^m t_2^n w = 0$ for n sufficiently large and $E_{r,s}w = 0$ for s sufficiently large, or equivalently, if $T_m(x) \in \mathcal{E}(W)$ and $E_r(x) \in \mathcal{E}(W)$ for $m, r \in \mathbb{Z}$. We say an $\mathcal{L}^*$-module W is of* level ℓ_{1234} *if the central element K_i acts as scalar ℓ_i for $i = 1,2,3,4$.*

Next we associate the Lie algebra $\mathcal{L}^*$ with a specific vertex algebra and its ϕ-coordinated modules. We first construct a new Lie algebra.

Definition 4.3. *Let $\widehat{\mathfrak{L}^*}$ be a vector space spanned by $T^m \otimes t^n, E^r \otimes t^s, K_i$ for $(m,n),(r,s) \in \mathbb{Z}^2\backslash\{(0,0)\}, i = 1,2,3,4$, we define*

$$\begin{aligned}
&[T^m \otimes t^n, T^r \otimes t^s] = 0; \quad [K_i, \widehat{\mathfrak{L}^*}] = 0, \;\; \textit{for} \;\; i = 1,2,3,4;\\
&[T^m \otimes t^n, E^r \otimes t^s]\\
&\quad = (nr - ms)T^{m+r} \otimes t^{n+s-1} + m\delta_{m+r,0}\delta_{n+s+1,0}K_1 + n\delta_{m+r,0}\delta_{n+s,0}K_2;\\
&[E^m \otimes t^n, E^r \otimes t^s]\\
&\quad = (nr - ms)E^{m+r} \otimes t^{n+s-1} + m\delta_{m+r,0}\delta_{n+s+1,0}K_3 + n\delta_{m+r,0}\delta_{n+s,0}K_4.
\end{aligned}$$

It is straightforward to check that these are Lie brackets, so $\widehat{\mathfrak{L}^*}$ is a Lie algebra. For $(m,n),(r,s) \in \mathbb{Z}^2\backslash\{(0,0)\}$, we denote $T^m \otimes t^n, E^r \otimes t^s$

by $(T^m)_n, (E^r)_s$, and set

$$T^m(x) = \sum_{n\in\mathbb{Z}} (T^m)_n x^{-n-1} \quad E^r(x) = \sum_{s\in\mathbb{Z}} (E^r)_s x^{-s-1}. \tag{4.4}$$

Then the defining relations of $\widehat{\mathfrak{L}^*}$ amount to:

$$\begin{aligned}
&[T^m(x_1), E^r(x_2)]\\
&\quad = (m+r)T^{m+r}(x_2)\frac{\partial}{\partial x_2}x_1^{-1}\delta\left(\frac{x_2}{x_1}\right)\\
&\qquad + m\left(\frac{\partial}{\partial x_2}T^{m+r}(x_2)\right)x_1^{-1}\delta\left(\frac{x_2}{x_1}\right)\\
&\qquad + m\delta_{m+r,0}x_1^{-1}\delta\left(\frac{x_2}{x_1}\right)K_1 + \delta_{m+r,0}\frac{\partial}{\partial x_2}x_1^{-1}\delta\left(\frac{x_2}{x_1}\right)K_2,
\end{aligned} \tag{4.5}$$

and

$$\begin{aligned}
&[E^m(x_1), E^r(x_2)]\\
&\quad = (m+r)E^{m+r}(x_2)\frac{\partial}{\partial x_2}x_1^{-1}\delta\left(\frac{x_2}{x_1}\right)\\
&\qquad + m\left(\frac{\partial}{\partial x_2}E^{m+r}(x_2)\right)x_1^{-1}\delta\left(\frac{x_2}{x_1}\right)\\
&\qquad + m\delta_{m+r,0}x_1^{-1}\delta\left(\frac{x_2}{x_1}\right)K_3 + \delta_{m+r,0}\frac{\partial}{\partial x_2}x_1^{-1}\delta\left(\frac{x_2}{x_1}\right)K_4.
\end{aligned} \tag{4.6}$$

Set

$$\begin{aligned}
\widehat{\mathfrak{L}^*}_{\geq 0} &= \coprod_{m\in\mathbb{Z}} (T^m\otimes\mathbb{C}[t]) \oplus \coprod_{r\in\mathbb{Z}} (E^r\otimes\mathbb{C}[t]) \oplus \sum_{i=1}^{4}\mathbb{C}K_i,\\
\widehat{\mathfrak{L}^*}_{<0} &= \coprod_{m\in\mathbb{Z}} \left(T^m\otimes t^{-1}\mathbb{C}[t^{-1}]\right) \oplus \coprod_{r\in\mathbb{Z}} \left(E^r\otimes t^{-1}\mathbb{C}[t^{-1}]\right).
\end{aligned}$$

We see that $\widehat{\mathfrak{L}^*}_{\geq 0}$ and $\widehat{\mathfrak{L}^*}_{<0}$ are Lie subalgebras and $\widehat{\mathfrak{L}^*} = \widehat{\mathfrak{L}^*}_{\geq 0} \oplus \widehat{\mathfrak{L}^*}_{<0}$ as a vector space. Let $\ell_i \in \mathbb{C}, i = 1,2,3,4$, we denote by $\mathbb{C}_{\ell_{1234}} = \mathbb{C}$ the one-dimensional $\widehat{\mathfrak{L}^*}_{\geq 0}$-module with $\coprod_{m\in\mathbb{Z}}(T^m\otimes\mathbb{C}[t]) \oplus \coprod_{r\in\mathbb{Z}}(E^r\otimes\mathbb{C}[t])$ acting trivially and K_i acting as ℓ_i for $i = 1,2,3,4$. Form the induced module

$$V_{\widehat{\mathfrak{L}^*}}(\ell_{1234}, 0) = U(\widehat{\mathfrak{L}^*}) \otimes_{U(\widehat{\mathfrak{L}^*}_{\geq 0})} \mathbb{C}_{\ell_{1234}}.$$

Set $\mathbf{1} = 1\otimes 1 \in V_{\widehat{\mathfrak{L}^*}}(\ell_{1234}, 0)$, define a linear operator $\overline{d}$ on $\widehat{\mathfrak{L}^*}$ by

$$\overline{d}(K_i) = 0, \text{ for } i = 1,2,3,4,$$
$$\overline{d}(T^m\otimes t^n) = -nT^m\otimes t^{n-1}, \text{ and } \overline{d}(E^r\otimes t^s) = -sE^r\otimes t^{s-1}.$$

By Theorem 5.7.1 of [24], $V_{\widehat{\mathfrak{L}^*}}(\ell_{1234}, 0)$ is a vertex algebra, which is uniquely determined by the condition that $\mathbf{1}$ is the vacuum vector and

$$Y((T^m)_{-1}\mathbf{1}, x) = T^m(x), \tag{4.7}$$

$$Y((E^r)_{-1}\mathbf{1},x)=E^r(x) \tag{4.8}$$

for $(T^m)_{-1},(E^r)_{-1}\in\widehat{\mathfrak{L}^*}$, $m,r\in\mathbb{Z}$. Furthermore, $\{(T^m)_{-1}\mathbf{1},(E^r)_{-1}\mathbf{1}\mid m,r\in\mathbb{Z}\}$ is a generating subset of $V_{\widehat{\mathfrak{L}^*}}(\ell_{1234},0)$.

Similarly, $V_{\widehat{\mathfrak{L}^*}}(\ell_{1234},0)$ has its universal property like Remark 2.7 and Remark 2.11. As one of our main results in this section, we have the following theorem.

Theorem 4.4. *Let W be a restricted $\mathcal{L}^*$-module of level ℓ_{1234}. Then there exists a ϕ-coordinated $V_{\widehat{\mathfrak{L}^*}}(\ell_{1234},0)$-module structure $Y_W(\cdot,x)$ on W, which is uniquely determined by*

$$Y_W((T^m)_{-1}\mathbf{1},x)=T_m(x)\ \ \textit{and}\ \ Y_W((E^m)_{-1}\mathbf{1},x)=E_m(x)\qquad \textit{for } m\in\mathbb{Z}.$$

Proof. Since $\{(T^m)_{-1}\mathbf{1},(E^r)_{-1}\mathbf{1}\mid m,r\in\mathbb{Z}\}$ generates $V_{\widehat{\mathfrak{L}^*}}(\ell_{1234},0)$ as a vertex algebra, the uniqueness is clear. We now prove the existence. Set $U_W=\{\mathbf{1}_W\}\cup\{T_m(x),E_m(x)\mid m\in\mathbb{Z}\}\subset\mathcal{E}(W)$. From (4.2) and (4.3), by using Lemma 2.1 of [29] we see that

$$(x_1-x_2)^2[T_m(x_1),E_r(x_2)]=0\ \ \text{and}\ \ (x_1-x_2)^2[E_m(x_1),E_r(x_2)]=0. \tag{4.9}$$

Then U_W is a local subset of $\mathcal{E}(W)$. By Theorem 2.3, U_W generates a vertex algebra $\langle U_W\rangle_e$ under the vertex operator operation $Y^e_{\mathcal{E}}$ with W a ϕ-coordinated module, where $Y_W(a(x),z)=a(z)$ for $a(x)\in\langle U_W\rangle_e$. With (4.2) and (4.3), by using Lemma 4.13 or Proposition 4.14 of [29], we have

$$\begin{aligned}
&T_m(x)^e_iE_r(x)=0\ \ \text{for } i\geq 2,\\
&T_m(x)^e_1E_r(x)=(m+r)T_{m+r}(x)+\delta_{m+r,0}\ell_2\mathbf{1}_W,\\
&T_m(x)^e_0E_r(x)=m\left(x\frac{\partial}{\partial x}T_{m+r}(x)\right)+m\delta_{m+r,0}\ell_1\mathbf{1}_W,
\end{aligned}$$

and

$$\begin{aligned}
&E_m(x)^e_iE_r(x)=0\ \ \text{for } i\geq 2,\\
&E_m(x)^e_1E_r(x)=(m+r)E_{m+r}(x)+\delta_{m+r,0}\ell_4\mathbf{1}_W,\\
&E_m(x)^e_0E_r(x)=m\left(x\frac{\partial}{\partial x}E_{m+r}(x)\right)+m\delta_{m+r,0}\ell_3\mathbf{1}_W.
\end{aligned}$$

Then again by Borcherds' commutator formula we have

$$\begin{aligned}
&[Y^e_{\mathcal{E}}(T_m(x),x_1),Y^e_{\mathcal{E}}(E_r(x),x_2)]\\
&=\sum_{i\geq 0}Y^e_{\mathcal{E}}(T_m(x)^e_iE_r(x),x_2)\frac{1}{i!}\left(\frac{\partial}{\partial x_2}\right)^i x_1^{-1}\delta\left(\frac{x_2}{x_1}\right)\\
&=mY^e_{\mathcal{E}}(x\frac{\partial}{\partial x}T_{m+r}(x),x_2)x_1^{-1}\delta\left(\frac{x_2}{x_1}\right)+m\delta_{m+r,0}\ell_1\mathbf{1}_Wx_1^{-1}\delta\left(\frac{x_2}{x_1}\right)\\
&\quad+(m+r)Y^e_{\mathcal{E}}(T_{m+r}(x),x_2)\frac{\partial}{\partial x_2}x_1^{-1}\delta\left(\frac{x_2}{x_1}\right)+\delta_{m+r,0}\ell_2\mathbf{1}_W\frac{\partial}{\partial x_2}x_1^{-1}\delta\left(\frac{x_2}{x_1}\right),
\end{aligned}$$

and

$$[Y_{\mathcal{E}}^e(E_m(x),x_1),Y_{\mathcal{E}}^e(E_r(x),x_2)]$$
$$=\sum_{i\geq 0}Y_{\mathcal{E}}^e(E_m(x)_i^e E_r(x),x_2)\frac{1}{i!}\left(\frac{\partial}{\partial x_2}\right)^i x_1^{-1}\delta\left(\frac{x_2}{x_1}\right)$$
$$=mY_{\mathcal{E}}^e(x\frac{\partial}{\partial x}E_{m+r}(x),x_2)x_1^{-1}\delta\left(\frac{x_2}{x_1}\right)+m\delta_{m+r,0}\ell_3\mathbf{1}_W x_1^{-1}\delta\left(\frac{x_2}{x_1}\right)$$
$$+(m+r)Y_{\mathcal{E}}^e(E_{m+r}(x),x_2)\frac{\partial}{\partial x_2}x_1^{-1}\delta\left(\frac{x_2}{x_1}\right)+\delta_{m+r,0}\ell_4\mathbf{1}_W\frac{\partial}{\partial x_2}x_1^{-1}\delta\left(\frac{x_2}{x_1}\right).$$

Similar as in the proof of Theorem 2.12 we get that W is a ϕ-coordinated module with

$$Y_W((T^m)_{-1}\mathbf{1},x)=T_m(x)\ \text{ and }\ Y_W((E^m)_{-1}\mathbf{1},x)=E_m(x)\quad \text{for } m\in\mathbb{Z}.$$

∎

On the other hand, we have the following statement.

Theorem 4.5. *Let W be a ϕ-coordinated $V_{\widehat{\mathfrak{L}^*}}(\ell_{1234},0)$-module. Then W is a restricted $\mathcal{L}^*$-module of level ℓ_{1234} with $T_m(x)=Y_W((T^m)_{-1}\mathbf{1},x)$, and $E_m(x)=Y_W((E^m)_{-1}\mathbf{1},x)$ for $m\in\mathbb{Z}$.*

Proof. For $m,r\in\mathbb{Z}$, since $Y((T^m)_{-1}\mathbf{1},x)=T^m(x),Y((E^m)_{-1}\mathbf{1},x)=E^m(x)$, with (4.5) and (4.6), by using (2.1) we see that

$$(x_1-x_2)^2[Y((T^m)_{-1}\mathbf{1},x_1),Y((E^r)_{-1}\mathbf{1},x_2)]=0,$$
$$(x_1-x_2)^2[Y((E^m)_{-1}\mathbf{1},x_1),Y((E^r)_{-1}\mathbf{1},x_2)]=0.$$

Note that for $m,r\in\mathbb{Z},i\geq 0$

$$((T^m)_{-1}\mathbf{1})_i(E^r)_{-1}\mathbf{1}=(T^m)_i(E^r)_{-1}\mathbf{1}=[T^m\otimes t^i,E^r\otimes t^{-1}]\mathbf{1}$$
$$=(ri+m)T^{m+r}\otimes t^{i-2}\mathbf{1}+m\delta_{m+r,0}\delta_{i,0}\ell_1\mathbf{1}+i\delta_{m+r,0}\delta_{i-1,0}\ell_2\mathbf{1}.$$

and

$$((E^m)_{-1}\mathbf{1})_i(E^r)_{-1}\mathbf{1}=(E^m)_i(E^r)_{-1}\mathbf{1}=[E^m\otimes t^i,E^r\otimes t^{-1}]\mathbf{1}$$
$$=(ri+m)E^{m+r}\otimes t^{i-2}\mathbf{1}+m\delta_{m+r,0}\delta_{i,0}\ell_3\mathbf{1}+i\delta_{m+r,0}\delta_{i-1,0}\ell_4\mathbf{1}.$$

By Proposition 5.9 of [28], we have

$$[Y_W((T^m)_{-1}\mathbf{1},x_1),Y_W((E^r)_{-1}\mathbf{1},x_2)]$$
$$=\mathrm{Res}_{x_0}x_1^{-1}\delta\left(\frac{x_2e^{x_0}}{x_1}\right)x_2e^{x_0}Y_W(Y(T^m,x_0)E^r,x_2)$$
$$=mY_W((T^{m+r})_{-2}\mathbf{1},x_2)\delta\left(\frac{x_2}{x_1}\right)+m\delta_{m+r,0}\delta\left(\frac{x_2}{x_1}\right)\ell_1\mathbf{1}_W$$
$$+(m+r)Y_W((T^{m+r})_{-1}\mathbf{1},x_2)\left(x_2\frac{\partial}{\partial x_2}\right)\delta\left(\frac{x_2}{x_1}\right)$$

$$+\delta_{m+r,0}\left(x_2\frac{\partial}{\partial x_2}\right)\delta\left(\frac{x_2}{x_1}\right)\ell_2\mathbf{1}_W,$$

and

$$\begin{aligned}&[Y_W((E^m)_{-1}\mathbf{1},x_1),Y_W((E^r)_{-1}\mathbf{1},x_2)]\\&=\mathrm{Res}_{x_0}x_1^{-1}\delta\left(\frac{x_2e^{x_0}}{x_1}\right)x_2e^{x_0}Y_W(Y(E^m,x_0)E^r,x_2)\\&=mY_W((E^{m+r})_{-2}\mathbf{1},x_2)\delta\left(\frac{x_2}{x_1}\right)+m\delta_{m+r,0}\delta\left(\frac{x_2}{x_1}\right)\ell_3\mathbf{1}_W\\&\quad+(m+r)Y_W((E^{m+r})_{-1}\mathbf{1},x_2)\left(x_2\frac{\partial}{\partial x_2}\right)\delta\left(\frac{x_2}{x_1}\right)\\&\quad+\delta_{m+r,0}\left(x_2\frac{\partial}{\partial x_2}\right)\delta\left(\frac{x_2}{x_1}\right)\ell_4\mathbf{1}_W,\end{aligned}$$

then we can prove as Theorem 2.13 that W is a restricted $\mathcal{L}^*$-module of level ℓ_{1234} with $T_m(x)=Y_W((T^m)_{-1}\mathbf{1},x)$, and $E_m(x)=Y_W((E^m)_{-1}\mathbf{1},x)$ for $m\in\mathbb{Z}$. □

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References

[1] T. Abe, G. Buhl and C. Dong, *Rationality, regularity, and C_2-cofiniteness*, Trans. Amer. Math. Soc. **356** (2004), 3391–3402.
[2] E. Arbarello, C. De Concini, V. Kac and C. Procesi, *Moduli spaces of curves and representation theory*, Comm. Math. Phys. **117** (1988), 1–36.
[3] D. Adamović and G. Radobolja, *Free field realization of the twisted Heisenberg-Virasoro algebra at level zero and its applications*, J. Pure and Appl. Algebra **219** (2015), 4322–4342.
[4] D. Adamović and G. Radobolja, *Self-dual and logarithmic representations of the twisted Heisenberg-Virasoro algebra at level zero*, Commun. Contemp. Math. **21** (2019), 1850008, 26pp.
[5] Y. Billig, *Representations of the twisted Heisenberg-Virasoro algebra at level zero*, Canad. Math. Bull. **46** (2003), 529–537.
[6] Y. Billig, *A category of modules for the full toroidal Lie algebra*, Int. Math. Res. Not. **2006**, Art. ID. 68395, 46pp.
[7] S. Berman, Y. Billig and J. Szmigielski, *Vertex operator algebras and the representation theory of toroidal algebras*, Contemporary Math. **297**, Amer. Math. Soc., Providence, 2002, 1–26.
[8] C. Dong and J. Lepowsky, Generalized vertex algebras and relative vertex operators, Birkhauser Boston, Inc., Boston, 1993.
[9] C. Dong and X.Lin, *Unitary vertex operator algebras*, J. Algebra **397** (2014), 252–277.
[10] C. Dong and Q. Wang, *The structure of parafermion vertex operator algebras*, Commun. Math. Phys. **299** (2010), 783–792.
[11] C. Dong, H. Li and G. Mason, *Regularity of rational vertex operator algebra*, Adv. Math. **312** (1997), 148–166.

[12] C. Dong, H. Li and G. Mason, *Twisted representations of vertex operator algebras*, Math. Ann. **310** (1998), 571–600.

[13] I. Frenkel, Y. Huang and J. Lepowsky, *On axiomatic approaches to vertex operator algebras and modules*, Mem. Amer. Math. Soc. **104** (1993), no. 494, Viii+64 pp.

[14] I. Frenkel, J. Lepowsky and A. Meurman, Vertex operator algebras and the Monster, Academic Press, Inc., Boston, 1988.

[15] I. B. Frenkel and Y. Zhu, *Vertex operator algebras associated to representations of affine and Virasoro algebras*, Duke Math. J. **66** (1992), 123–168.

[16] M. Golenishcheva-Kutuzova and V. Kac, Γ-*conformal algebras*, J. Math. Phys. **39** (1998), 2290–2305.

[17] P. Goddard, A. Kent and D. Olive, *Virasoro algebras and coset space models*, Phys. Lett. B **152** (1985), 88–92.

[18] H. Guo, H. Li, S. Tan and Q. Wang, *q-Virasoro algebra and vertex algebras*, J. Pure Appl. Algebra **219** (2015), 1258–1277.

[19] H. Guo and Q. Wang, *Associating vertex algebras to the unitary Lie algebra*, J. Algebra **424** (2015), 126–146.

[20] C. Jiang and H. Li, *Associating quantum vertex algebras to Lie algebra $\mathfrak{gl}_\infty$*, J. Algebra **399** (2014), 1086–1106.

[21] C. Jiang and Z. Lin, *The commutant of $L_{\widehat{sl_2}}(n,0)$ in the vertex operator algebra $L_{\widehat{sl_2}}(1,0)^{\otimes n}$*, Adv. Math. **301** (2016), 227–257.

[22] V. Kac and D. Peterson, *Infinite-dimensional Lie algebra, theta functions and modular forms*, Adv. Math. **53** (1984), 125–264.

[23] B. Lian and A. Linshaw, *Howe pairs in the theory of vertex algebras*, J. Algebra **317** (2007), 111–152.

[24] J. Lepowsky and H. Li, Introduction to vertex operator algebras and their representations, Birkhäuser, Inc., Boston, 2004.

[25] H. Li, *A new construction of vertex algebras and quasi modules for vertex algebras*, Adv. Math. **202** (2006), 232–286.

[26] H. Li, *On certain generalizations of twisted affine Lie algebras and quasimodules for Γ-vertex algebras*, J. Pure Appl. Algebra **209** (2007), 853–871.

[27] H. Li, *Local systems of vertex operators, vertex superalgebras and modules*, J. Pure Appl. Algebra **109** (1996), 143–195.

[28] H. Li, *ϕ-coordinated quasi-modules for quantum vertex algebras*, Comm. Math. Phys. **308** (2011), 703–741.

[29] H. Li, *G-equivariant ϕ-coordinated quasi-modules for quantum vertex algebras*, J. Math. Phys. **54** (2013), 051704, 26 pp.

[30] H. Li, *Nonlocal vertex algebras generated by formal vertex operators*, Selecta Math. (N.S.) **11** (2005), 349–397.

[31] H. Li, *Some finiteness properties of regular vertex operator algebras*, J. Algebra **212** (1999), 495–514.

[32] H. Li, S. Tan and Q. Wang, *Toroidal vertex algebras and their modules*, J. Algebra **365** (2012) 50–82.

[33] M. Xue, W. Lin and S. Tan, *Central extension, derivations and automorphism group for Lie algebras arising from 2-dimensional torus*, J. Lie Theory **16** (2006), 139–153.

[34] V. Kac and Rehana, Bombay lectures on highest weight representation of infinite dimensional Lie algebras, World Scientific Publishing Co., Inc., Teaneck, 1987.

[35] W. Wang, *Rationality of Virasoro vertex operator algebras*, Internat. Math. Res. Notices **1993**, 197–211.

[36] Y. Zhu, *Modular invariance of characters of vertex operator algebras*, J. Amer. Math. Soc **9** (1996), 237–302.

H. Guo
School of Mathematics and Statistics
Central China Normal University
Wuhan 430079
China

Q. Wang
School of Mathematical Sciences
Xiamen University
Xiamen 361005
China

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